Appendix A

Evaluation of an Important Integral

Our goal is to show that

$$\int_0^\infty e^{-a^2 x^2}\, dx = \frac{\sqrt{\pi}}{2a} \qquad \text{(A.1)}$$

where a is a positive constant. But first we will prove the slightly simpler case

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2} \qquad \text{(A.2)}$$

and then use a standard calculus technique to get Equation A.1 from Equation A.2.

To obtain Equation A.2 is not an easy task, because the standard methods of evaluating integrals don't work. In our method, we integrate the function

$$f(x, y) = e^{-x^2-y^2} \qquad \text{(A.3)}$$

over the infinite region R of the first quadrant of the xy plane, as shown in Figure A.1, by two different, but equivalent, ways; that is, we want to evaluate

$$\int_R e^{-x^2-y^2}\, dA \qquad \text{(A.4)}$$

by two different methods.

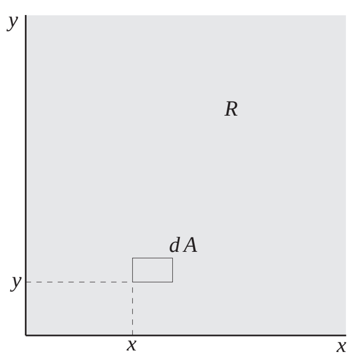

Figure A.1

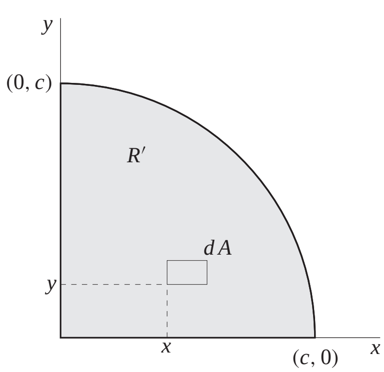

Figure A.2

In the first method, to perform the integration over R, we choose a quartercircle of radius c that encloses a region R', as shown in Figure A.2, and then let $c \to \infty$; thus

$$\int_R e^{-x^2-y^2}\, dA = \lim_{c\to\infty} \int_{R'} e^{-x^2-y^2}\, dA \qquad \text{(A.5)}$$

To evaluate the integral over R', the symmetry of a circle suggests that we use polar coordinates, as illustrated in Figure A.3. By inspection, we see that $dA = r\,d\phi\,dr$, and also

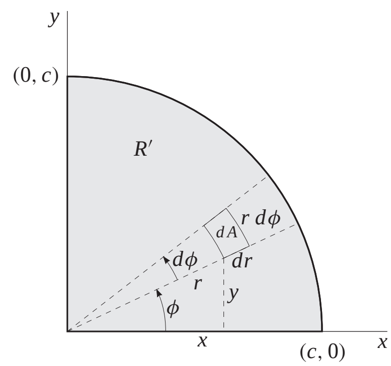

Figure A.3

that $x = r\cos\phi$ and $y = r\sin\phi$; we quickly read that r runs from 0 to ∞ and ϕ from 0 to $\pi/2$. Thus, substituting and manipulating, we have

$$\begin{aligned}
\int_{R'} e^{-x^2-y^2}\, dA &= \int_0^c \int_0^{\pi/2} e^{(-r^2\cos^2\phi - r^2\sin^2\phi)}\,(r\,d\phi\,dr) \\
&= \int_0^c \int_0^{\pi/2} r e^{-r^2}\, d\phi\,dr \\
&= \int_0^c r e^{-r^2} \left(\int_0^{\pi/2} d\phi\right) dr \\
&= \frac{\pi}{2}\int_0^c e^{-r^2} r\,dr \qquad \text{(A.6)}
\end{aligned}$$

To determine the integral in Equation A.6, we change the variable of integration by defining $r' = -r^2$, then $dr' = -2r\,dr$, and the limits on r' run from 0 to $-c^2$; thus we have

$$\begin{aligned}
\frac{\pi}{2}\int_0^c e^{-r^2} r\,dr &= \left(\frac{\pi}{2}\right)\left(-\frac{1}{2}\right)\int_0^{-c^2} e^{r'}\, dr' \\
&= -\frac{\pi}{4}\left[e^{r'}\right]_0^{-c^2} \\
&= \frac{\pi}{4}\left(1 - e^{-c^2}\right) \qquad \text{(A.7)}
\end{aligned}$$

We now use Equations A.6 and A.7 in Equation A.5 to obtain the integral of $e^{-x^2-y^2}$ over the region R:

$$\begin{aligned}\int_R e^{-x^2-y^2}\,dA &= \lim_{c\to\infty}\int_{R'} e^{-x^2-y^2}\,dA\\ &= \lim_{c\to\infty}\left[\frac{\pi}{4}\left(1-e^{-c^2}\right)\right]\\ &= \frac{\pi}{4}\end{aligned}\tag{A.8}$$

Thus, by our first method, we have found the very simple value of $\pi/4$ for this integral over the infinite region R (see Figure A.1). We now want to evaluate the integral in A.4 by another method.

As a start to the second method, we recognize that there are many ways to cover the infinite region R by going to a limit of some starting region, it does not have to be a quarter-circle region like that in Figure A.2; we find that the square region R'' shown in Figure A.4 does the job. We begin with an equation like that of Equation A.5:

$$\int_R e^{-x^2-y^2}\,dA = \lim_{c\to\infty}\int_{R''} e^{-x^2-y^2}\,dA \tag{A.9}$$

But now the right side of Equation A.9 says that we must evaluate $e^{-x^2-y^2}$ over the square region R''. The square symmetry suggests that we choose a rectangular area dA with sides dx and dy (see Figure A.4); we then proceed as follows:

$$\begin{aligned}\int_{R''} e^{-x^2-y^2}\,dA &= \int_0^c\int_0^c e^{-x^2-y^2}\,dx\,dy\\ &= \int_0^c\int_0^c e^{-x^2}e^{-y^2}\,dx\,dy\\ &= \int_0^c e^{-y^2}\left(\int_0^c e^{-x^2}\,dx\right)dy\\ &= \left(\int_0^c e^{-x^2}\,dx\right)\left(\int_0^c e^{-y^2}\,dy\right)\\ &= \left(\int_0^c e^{-x^2}\,dx\right)^2\end{aligned}\tag{A.10}$$

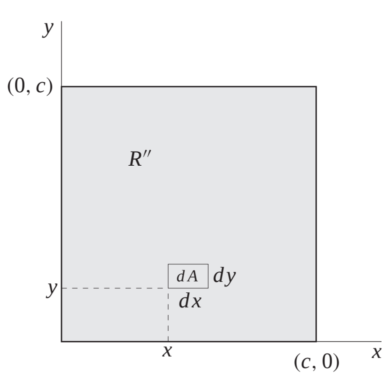

Figure A.4

where in the last step of Equation A.10 we used the fact that both x and y are dummy variables; that is, the symbol we use for the variable of integration does not affect the value of a definite integral. Thus, we change the y variable to x to make a simpler expression, as well as an expression that is exactly what we want. We now substitute Equation A.10 into Equation A.9 and obtain the result of our second method:

$$\begin{aligned}\int_R e^{-x^2-y^2}\,dA &= \lim_{c\to\infty}\int_{R''} e^{-x^2-y^2}\,dA\\ &= \lim_{c\to\infty}\left(\int_0^c e^{-x^2}\,dx\right)^2\\ &= \left(\int_0^\infty e^{-x^2}\,dx\right)^2\end{aligned}\tag{A.11}$$

We observe that Equations A.8 and A.11 give two different answers for the integral of $e^{-x^2-y^2}$ over the region R. However, these results must be equal, and therefore

$$\int_R e^{-x^2-y^2}\,dA = \left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \frac{\pi}{4} \tag{A.12}$$

Finally, taking the square root, we get

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \tag{A.13}$$

and we have obtained Equation A.2.

But our ultimate goal is to get Equation A.1. So we define a new variable of integration $x' = ax$, which gives $dx' = a\,dx$. Substituting this new variable information into Equation A.1 yields

$$\begin{aligned}\int_0^\infty e^{-a^2x^2}\,dx &= \frac{1}{a}\int_0^\infty e^{-x'^2}\,dx'\\ &= \frac{1}{a}\,\frac{\sqrt{\pi}}{2}\\ &= \frac{\sqrt{\pi}}{2a}\end{aligned}\tag{A.14}$$

where we have used Equation A.13 to evaluate the second integral; thus, we have obtained our desired result.