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PART II

PHYSICAL OR WAVE OPTICS

In geometrical optics, we used the ray model to represent the behavior of light, which included the laws of reflection and refraction. The only hint that a more comprehensive model was needed occurred when we talked about color in terms of wavelength. This more inclusive model is called wave optics, or by the older name, physical optics, where physical is the classical term that refers to matter or the material world. Explicitly, here in Part II, we now model light as a wave that travels through space or matter.

Whether light behaved as a stream of particles or waves has an interesting history. Newton argued for the particle model, but then experiments were performed that more easily explained in terms of the wave model. Now we know that light is both, it has both wave-like and particle-like properties. In the chapters that follow, we describe the wave-like properties, which ultimately are a part of electromagnetic theory devised by Maxwell in latter part of the 19th century.

The first experiments that required the wave theory for explanation occurred in the early part of the 19th century. These experiments showed that light could produce interference patterns, regions of alternating dark and bright areas. But the vibration that composes a wave can be longitudinal, a vibration in the same direction as wave travel (like sound waves), or transverse, a vibration perpendicular to the direction of travel. Eventually, it was concluded that light behaved as a transverse wave. Then, with the advent of electromagnetic theory, the vibration was envisioned as an electric vector and a magnetic vector.

We describe waves in terms of cosine and sine functions, which means that trigonometry is important. A big help in performing the mathematics of these functions is provided by the exponential function in the complex plane for it combines in one function both the cosine and the sine properties. Although the mathematics is more abstract, the manipulation is usually much simpler. We start our discussion on wave optics by developing the mathematics in the complex plane. Then we review the properties of waves, and show how to describe waves in the complex plane.

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Chapter 5

Complex Algebra and Harmonic Waves

5.1 Introduction

Many of the properties of light are modeled in terms of sinusoidal or harmonic waves. By a harmonic wave we mean a sine function, a cosine function, or one of these functions with a phase constant; that is, we use the name harmonic as a generic term. In general, these harmonic waves are traveling waves: their characteristics we shall describe later in the chapter. Because complex algebra provides an efficient means of manipulating expressions that contain harmonic waves, we start with a description of some of the features of complex algebra.

5.2 Complex Algebra

5.2.1 Representation of complex numbers

Complex algebra deals with complex numbers: they are represented in **rectangular form** or Cartesian form as

$$z = x + iy \tag{5.1}$$

where

$$\begin{aligned} x &= \text{ a real number called the real part of } z \\ &= \mathrm{Re}(z) \end{aligned} \tag{5.2a}$$

$$\begin{aligned} y &= \text{a real number called the imaginary part of } z \\ &= \mathrm{Im}(z) \end{aligned} \tag{5.2b}$$

$$i = \sqrt{-1} \tag{5.2c}$$

As an aid to understand the properties of complex numbers, we represent a complex number in a plane called the **complex plane** (also called an Argand diagram). Two ways of displaying a complex number in the complex plane are shown in Figure 5.1. As the diagrams show, we graph the real part x along the horizontal axis (the Re axis), and the imaginary part y along the vertical axis (the Im axis). In Figure 5.1 (a), we represent the complex number as a point; in Figure 5.1 (b), as a vector-like quantity called a **phasor**—thus, we have a phasor diagram for the complex number z. Phasors have some of the properties of vectors in that they add and subtract like the vectors used in mechanics; however, unlike vectors, phasors are also multiplied and divided. Just like vectors, the x and y are regarded as the components of the phasor z. The i is always inserted somewhere in the y expression; in our notation, we usually put it in front of the expression as in Equation 5.1.

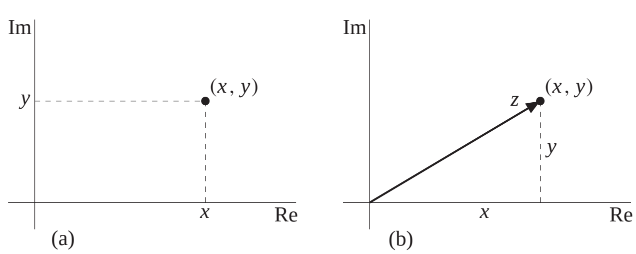

Figure 5.1

From Equation 5.2c, we get the important relations

$$i^2 = -1 \tag{5.3a}$$

$$i = -\frac{1}{i} \tag{5.3b}$$

and in general:

$$i^n = \begin{Bmatrix} i; \ n = 1, 5, 9, \ldots \\ -1; \ n = 2, 6, 10, \ldots \\ -i; \ n = 3, 7, 11, \ldots \\ 1; \ n = 4, 8, 12, \ldots \end{Bmatrix} \tag{5.4}$$

A complex number z also has a polar form representation, as we illustrate with the phasor diagram in Figure 5.2.

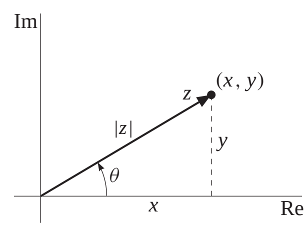

Figure 5.2

By inspection we read off

$$z = x + iy = |z|\cos\theta + i|z|\sin\theta \tag{5.5a}$$

$$= |z|(\cos\theta + i\sin\theta) \tag{5.5b}$$

where

$$\begin{aligned} |z| = \mathrm{mag}(z) &= \text{the magnitude (or absolute value) of } z \\ &= \sqrt{x^2 + y^2} \end{aligned} \tag{5.6a}$$

$$\begin{aligned} \theta = \arg(z) &= \text{the angle the phasor makes with the positive Re axis (or with a line parallel to the Re axis)} \\ &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned} \tag{5.6b}$$

By default, we usually write θ so that $-\pi < \theta \leq \pi$; however, any multiple of 2π when added or subtracted is an acceptable value for θ.

We now want to develop a new relationship, which will allow us to set the sine/cosine expression of Equation 5.5b equal to a new and shorter expression that makes complex algebra so useful for working with harmonic expressions. We start with the power series for the exponential function e^x, where x is a real number:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \dots \quad (5.7)$$

This equation suggests that to define the exponential function in the complex plane, we should replace x by the complex number z; thus

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \frac{z^7}{7!} + \dots \quad (5.8)$$

Now let's suppose that z equals the complex number $i\theta$, where θ is an angle measured in radians. Then Equation 5.8 becomes, with the help of Equation 5.4,

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} \\ &\qquad + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \dots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} \\ &\qquad + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \dots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + - \dots \\ &\qquad + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + - \dots\right) \\ &= \cos\theta + i\sin\theta \end{aligned} \quad (5.9)$$

because the power series for $\cos\theta$ is

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + - \dots \quad (5.10a)$$

and the analogous one for $\sin\theta$ is

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + - \dots \quad (5.10b)$$

The result embodied in Equation 5.9, namely

$$e^{i\theta} = \cos\theta + i\sin\theta \quad (5.11)$$

is the famous equation called **Euler's identity**: This equation has played an important role in the development of mathematics. A related equation is quickly obtained from Euler's identity as follows:

$$\begin{aligned} e^{-i\theta} = e^{i(-\theta)} &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta - i\sin\theta \end{aligned} \quad (5.12)$$

Finally, with Euler's identity in Equation 5.11, we change Equation 5.5b to give the elegant and useful expression for a complex number z in **polar form** as

$$z = |z|\, e^{i\theta} \quad (5.13)$$

When adding and subtracting complex numbers, the rectangular form of z in Equation 5.1 is easiest to use; when multiplying and dividing, the polar form in Equation 5.13 is most convenient.

Example 5.2.1 Rectangular to polar form.
Suppose a complex number is given in rectangular form as (see Equation 5.1)

$$z = -2 - i3$$

To change to polar form we use Equations 5.6 to calculate

$$|z| = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (-3)^2} = 3.61$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-3}{-2}\right) = -2.16\,\text{rad} = -123.7^\circ$$

and then write z in polar form with Equation 5.13:

$$z = 3.61\, e^{i(-2.16)} = 3.61\, e^{i(-123.7^\circ)}$$

Note that for simplicity we do not write the rad unit (the radian) beside the -2.16 value in $e^{i(-2.16)}$ because the rad is supplementary unit (and can be dropped when no confusion arises); thus, we treat the -2.16 as a real number with no physical units (see Section 1.1.2). However, for clarity, we include the degree unit written as $^\circ$. Also, note that when we substitute the values into $\tan^{-1}(y/x)$ we include the signs (here, minus signs) so that we can determine the proper quadrant for θ. The phasor diagram for z is shown in Figure 5.3.

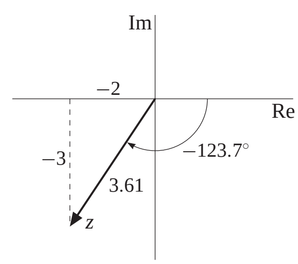

Figure 5.3

Example 5.2.2 Polar to rectangular form.
We start with a complex number in polar form:

$$z = 10\, e^{i30^\circ} = 10\, e^{i0.5236}$$

With the help of Equation 5.5a, we calculate

$$x = |z|\,\cos\theta = 10\,\cos 30^\circ = 8.66$$

$$y = |z|\,\sin\theta = 10\,\sin 30^\circ = 5$$

and substituting into Equation 5.1, we obtain z in rectangular or Cartesian form:

$$z = x + iy = 8.66 + i5$$

In Figure 5.4, we display the phasor diagram for z.

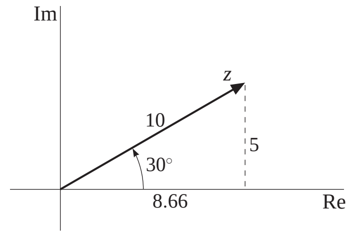

Figure 5.4

Example 5.2.3 Several important complex numbers. In rectangular and polar form, we have

$$z = 1 = e^{i0} \tag{5.14a}$$

$$z = -1 = 1\,e^{i\pi} = e^{i\pi} = e^{i180^\circ} \tag{5.14b}$$

with their phasor diagrams shown in Figure 5.5. For $z = -1$ in Figure 5.5(b), the angle is written as π rad or 180°; also, the -1 in the diagram represents the value of x when z is written in rectangular form—the 1 is the magnitude of z when written in polar form. It is interesting to note that when $-1 = e^{i\pi}$ is written as $e^{i\pi} + 1 = 0$, we have a relation that connects five of the fundamental numbers of mathematics.

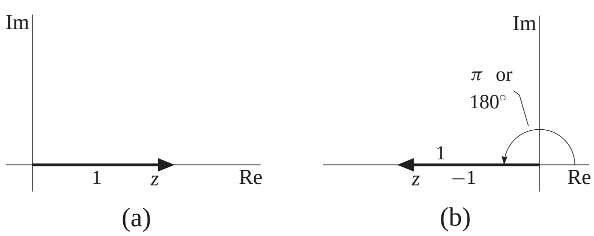

Figure 5.5

In a similar manner, we write

$$z = i = 1\,e^{i\pi/2} = e^{i\pi/2} = e^{i90^\circ} \tag{5.15a}$$

$$z = -i = 1\,e^{i(-\pi/2)} = e^{i(-\pi/2)} = e^{i(-90^\circ)} \tag{5.15b}$$

and show the phasor diagrams in Figure 5.6.

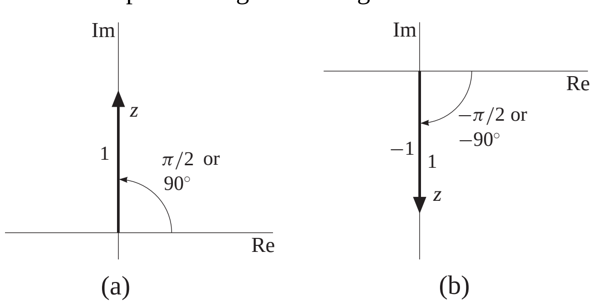

Figure 5.6

5.2.2 The negative of a complex number

If a complex number z in rectangular form is

$$z = x + iy \tag{5.16}$$

then the negative of z is defined as

$$-z = -x + i(-y) = -x - iy \tag{5.17}$$

In polar form, we have

$$-z = |z|\,e^{i(\theta\pm\pi)} = |z|\,e^{i(\theta\pm 180^\circ)} \tag{5.18}$$

We write $\pm\pi$ or $\pm 180^\circ$ in the argument to put the angle in the proper quadrant; that is, $-\pi < (\theta \pm \pi) \leq \pi$, or in terms of degrees, $-180^\circ < (\theta \pm 180^\circ) \leq 180^\circ$ (see the statement under Equation 5.6b). We illustrate with the phasor diagrams in Figure 5.7. The solid dots are used to mark the tails of the phasors at the origin.

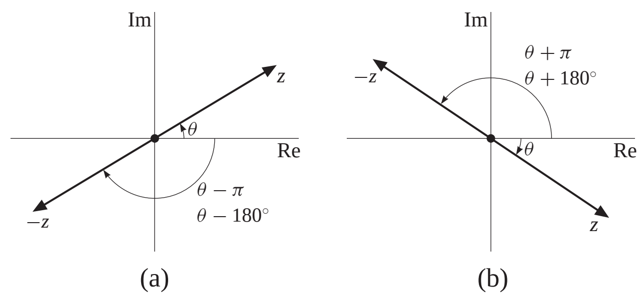

Figure 5.7

5.2.3 Addition of complex numbers

If z_1 and z_2 are complex numbers in rectangular form, namely,

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2$$

then addition is defined as

$$z = z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{5.19}$$

and is illustrated in the phasor diagrams of Figure 5.8. In

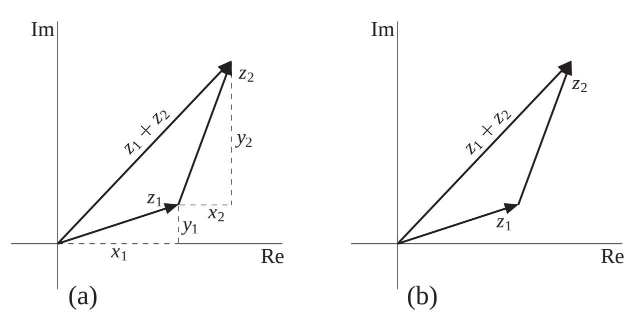

Figure 5.8 The (b) diagram is a simplified form of (a).

Figure 5.8(a), we include the real part x_1 and imaginary part y_1 of z_1, and similarly (x_2, y_2) for z_2, to better display how they determine the new complex number $z_1 + z_2$. In Figure 5.8(b), we simply draw the phasor diagram of the three complex numbers. We note how the phasor diagram for addition is a head-to-tail diagram just like that for vector addition. Also, we note by looking at z_2 that the tail of a phasor does not have to begin at the origin—it is permissible to move a phasor parallel to itself.

5.2.4 Subtraction of complex numbers

We define subtraction in terms of addition and the negative of a complex number; thus

$$z = z_1 - z_2 = z_1 + (-z_2) \tag{5.20}$$

The phasor diagram for subtraction is shown in Figure 5.9. The drawing in Figure 5.9(a) shows a parallelogram with the dashed phasors displaying $z_1 + (-z_2)$; the solid phasors show a quick way to think of $z_1 - z_2$, namely, that in subtraction the tails of the phasors touch with the phasor that represents $z_1 - z_2$ having its head touching the head of z_1. The phasor diagram in Figure 5.9(b) emphasizes this latter approach, and is the way we normally draw subtraction. As a check, we look at Figure 5.9(b) and note that the phasors z_2 and $z_1 - z_2$ have the head-to-tail property indicating that these phasors are added to give the sum z_1; that is,

$$z_2 + (z_1 - z_2) = z_2 + z_1 - z_2 = z_1$$

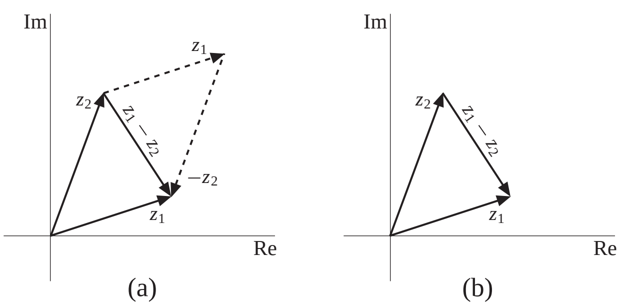

Figure 5.9 The (b) drawing is a shorthand form of (a).

Example 5.2.4 Addition and subtraction.
Suppose

$$z_1 = 2 + i3 \quad \text{and} \quad z_2 = 5 + i2$$

then

$$z_1 + z_2 = 7 + i5$$

and is diagrammed in Figure 5.10.

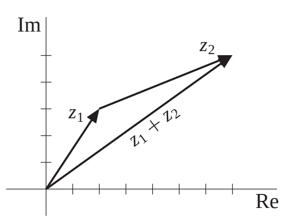

Figure 5.10

If we continue to let

$$z_1 = 2 + i3 \quad \text{and} \quad z_2 = 5 + i2$$

then we obtain two different answers by subtraction:

$$z_1 - z_2 = 2 + i3 - (5 + i2) = -3 + i$$

and

$$z_2 - z_1 = 5 + i2 - (2 + i3) = 3 - i$$

The phasor diagrams for $z_1 - z_2$ and $z_2 - z_1$ are shown in the drawings of Figure 5.11.

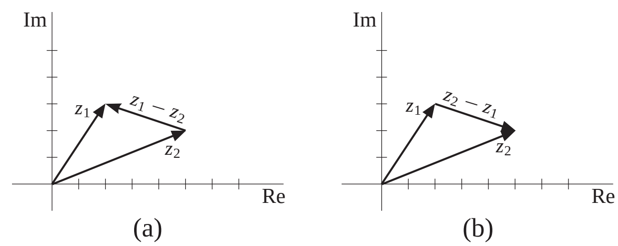

Figure 5.11

5.2.5 Multiplication of complex numbers

Complex numbers are most easily multiplied when they are in polar form, although it is possible to multiply them in rectangular form as well, as we discuss later in this section. The definition in polar form is

$$\begin{aligned} z = z_1 z_2 &= \left(|z_1|\, e^{i\theta_1}\right)\left(|z_2|\, e^{i\theta_2}\right) \\ &= |z_1||z_2|\, e^{i(\theta_1+\theta_2)} \end{aligned} \tag{5.21}$$

The phasor diagram in Figure 5.12 illustrates multiplication, where it is assumed that both $|z_1|$ and $|z_2|$ are greater than one. Except to observe how the angles add, there is no easy geometric interpretation for multiplication like there is for addition and subtraction.

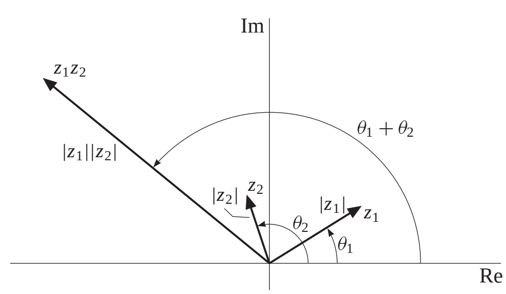

Figure 5.12

If z_1 and z_2 are given in rectangular form, then the expressions are multiplied term by term to obtain

$$\begin{aligned} z = z_1 z_2 &= (x_1 + i y_1)(x_2 + i y_2) \\ &= (x_1 x_2 - y_1 y_2) + i(x_2 y_1 + x_1 y_2) \end{aligned} \tag{5.22}$$

where we have used $i^2 = -1$.

We can obtain Equation 5.22 from Equation 5.21 by using Euler's identity in Equation 5.11 and some trig identities:

$$\begin{aligned} |z_1||z_2|\, e^{i(\theta_1+\theta_2)} &= |z_1||z_2|\, [\cos(\theta_1+\theta_2) + i \sin(\theta_1+\theta_2)] \\ &= |z_1||z_2|\, [\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2] \\ &\quad + i|z_1||z_2|\, [\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2] \\ &= |z_1||z_2| \left[\frac{x_1}{|z_1|}\frac{x_2}{|z_2|} - \frac{y_1}{|z_1|}\frac{y_2}{|z_2|} \right] \\ &\quad + i|z_1||z_2| \left[\frac{y_1}{|z_1|}\frac{x_2}{|z_2|} + \frac{x_1}{|z_1|}\frac{y_2}{|z_2|} \right] \\ &= (x_1 x_2 - y_1 y_2) + i(x_2 y_1 + x_1 y_2) \end{aligned} \tag{5.23}$$

5.2.6 Division of complex numbers

Division of complex numbers is also most easily defined in terms of the polar form:

$$z = \frac{z_1}{z_2} = \frac{|z_1|\, e^{i\theta_1}}{|z_2|\, e^{i\theta_2}} = \frac{|z_1|}{|z_2|}\, e^{i(\theta_1-\theta_2)} \tag{5.24}$$

To illustrate division, we draw the phasor diagram in Figure 5.13. For convenience, this diagram is based on the one for multiplication in Figure 5.12. Again, just like in multiplication, there is no easy geometrical interpretation.

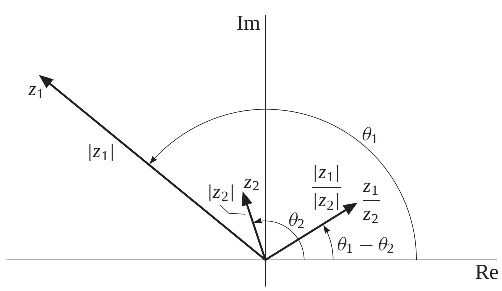

Figure 5.13

To obtain a rectangular form for division, we start with z_1 and z_2 in rectangular form and write

$$z = \frac{z_1}{z_2} = \frac{x_1 + i y_1}{x_2 + i y_2} \tag{5.25a}$$

To get z in rectangular form, we must remove the i in the denominator. A very useful procedure for manipulating complex number expressions of this type is to multiply the fraction by another fraction composed of the complex conjugate of the denominator; here, the fraction is $(x_2 - i y_2)/(x_2 - i y_2)$. Thus, starting with Equation 5.25a,

$$\begin{aligned} z &= \frac{x_1 + i y_1}{x_2 + i y_2}\,\frac{x_2 - i y_2}{x_2 - i y_2} \\ &= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \end{aligned} \tag{5.26b}$$

an expression that is certainly more complicated than the corresponding polar form one in Equation 5.24.

Equation 5.26b can also be obtained directly from the polar form in Equation 5.24 following the method used to obtain Equation 5.23. We leave this derivation as a problem for the reader.

5.2.7 The complex conjugate

Suppose z is a complex number given in rectangular or polar form as

$$z = x + i y = |z|\, e^{i\theta} \tag{5.27}$$

then the **complex conjugate** is named $z^\star$ and defined as

$$z^\star = x - i y = |z|\, e^{-i\theta} \tag{5.28}$$

In short, to turn a complex number into its conjugate simply replace i by $-i$; even if the complex number is given by a long expression, applying this rule works. We draw phasor diagrams to display the properties of $z, z^\star$ in Figure 5.14: in the (a) diagram, we represent the phasors in rectangular form, and in (b), polar form.

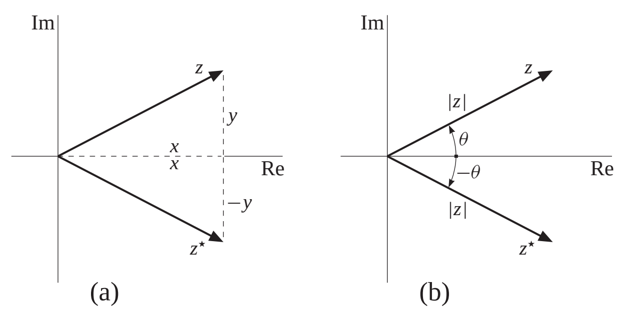

Figure 5.14

One of the results we quickly obtain from Equations 5.27 and 5.28 is that

$$z z^\star = |z|^2 \tag{5.29}$$

Two more important results are

$$\mathrm{Re}(z) = \frac{1}{2}(z + z^\star) \tag{5.30a}$$

$$\mathrm{Im}(z) = \frac{1}{2i}(z - z^\star) \tag{5.30b}$$

Equations 5.30 provide an analytical method for finding the real or imaginary part of a complex expression. We prove these equations by using the rectangular form as follows:

$$\frac{1}{2}(z+z^\star) = \frac{1}{2}(x+iy+x-iy) = x = \text{Re}(z)$$

$$\frac{1}{2i}(z-z^\star) = \frac{1}{2i}(x+iy-x+iy) = y = \text{Im}(z)$$

5.2.8 Summary

Addition and subtraction are most easily done when complex numbers or expressions are in rectangular form; multiplication and division when in polar form. In general, the complex conjugate helps to find the magnitude, the real part, and the imaginary part of a complex number or expression (see Equations 5.29 and 5.30).

Example 5.2.5 Multiplication.
Suppose

$$z_1 = 10e^{i\pi/6} \qquad \text{and} \qquad z_2 = 2e^{i\pi/18}$$

Then Equation 5.21 says that

$$z = z_1 z_2 = \left(10e^{i\pi/6}\right)\left(2e^{i\pi/18}\right) = 20e^{i2\pi/9}$$

Example 5.2.6 Division.
If

$$z_1 = 4e^{i\pi/3} \qquad \text{and} \qquad z_2 = 2e^{i\pi/15}$$

then Equation 5.24 gives

$$z = \frac{z_1}{z_2} = \frac{4e^{i\pi/3}}{2e^{i\pi/15}} = 2e^{i4\pi/15}$$

Example 5.2.7 The complex conjugate.
Suppose

$$z = \frac{1}{x+iy}$$

and we want to find $|z|^2$, Re(z), and Im(z). First, Equation 5.29 leads to

$$|z|^2 = zz^\star = \left(\frac{1}{x+iy}\right)\left(\frac{1}{x-iy}\right) = \frac{1}{x^2+y^2}$$

Then, Equation 5.30a gives

$$\text{Re}(z) = \frac{1}{2}(z+z^\star) = \frac{1}{2}\left(\frac{1}{x+iy} + \frac{1}{x-iy}\right)$$
$$= \frac{1}{2}\,\frac{x-iy+x+iy}{(x+iy)(x-iy)} = \frac{x}{x^2+y^2}$$

and in a similar way, Equation 5.30b yields

$$\text{Im}(z) = \frac{1}{2i}(z-z^\star) = \frac{1}{2i}\left(\frac{1}{x+iy} - \frac{1}{x-iy}\right)$$
$$= \frac{1}{2i}\,\frac{x-iy-x-iy}{(x+iy)(x-iy)} = -\frac{y}{x^2+y^2}$$

5.2.9 More properties of complex quantities

de Moivre's theorem. We obtain an important result, called de Moivre's (deh Mwahv′ ris) theorem, that is helpful in finding trig identities. We apply a rule of exponents to obtain

$$\left(e^{i\theta}\right)^n = e^{in\theta} \qquad (5.31a)$$

and then use Euler's identity in Equation 5.11 to get

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \qquad (5.31b)$$

where it is Equation 5.31b that states de Moivre's theorem. In practice, we usually apply this theorem with the left and right sides interchanged:

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

We illustrate with $n = 2$, but any value of n works. Thus,

$$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2$$
$$= \cos^2\theta - \sin^2\theta + i(2\sin\theta\cos\theta)$$

Equating the real and imaginary parts, we have the well-known identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

The conjugate of a conjugate. It is easy to prove that

$$\left(z^\star\right)^\star = z \qquad (5.32)$$

by using the rectangular form of a complex number (the polar form works just as well):

$$\left(z^\star\right)^\star = \left((x+iy)^\star\right)^\star = (x-iy)^\star = x+iy = z$$

and we have proved Equation 5.32.

The complex conjugate of a product. To prove that

$$(z_1 z_2)^\star = z_1^\star z_2^\star \tag{5.33}$$

it is easier to use the polar form; thus

$$\begin{aligned}(z_1 z_2)^\star &= \left(\left(|z_1| e^{i\theta_1}\right)\left(|z_2| e^{i\theta_2}\right)\right)^\star \\ &= \left(|z_1| e^{-i\theta_1}\right)\left(|z_2| e^{-i\theta_2}\right) = z_1^\star z_2^\star\end{aligned}$$

where we have found the complex conjugate by replacing i wherever it appeared by $-i$ (see Section 5.2.7).

The real part of a sum equals the sum of the real parts. In equation form, this important statement reads

$$\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2) \tag{5.34}$$

This equation includes subtraction because subtraction is defined in terms of addition (see Section 5.2.4). Using the rectangular form, we first evaluate the left side

$$\begin{aligned}\mathrm{Re}(z_1 + z_2) &= \mathrm{Re}(x_1 + iy_1 + x_2 + iy_2) \\ &= \mathrm{Re}[(x_1 + x_2) + i(y_1 + y_2)] \\ &= x_1 + x_2\end{aligned}$$

and then the right side

$$\begin{aligned}\mathrm{Re}(z_1) + \mathrm{Re}(z_2) &= \mathrm{Re}(x_1 + iy_1) + \mathrm{Re}(x_2 + iy_2) \\ &= x_1 + x_2\end{aligned}$$

Either way we have obtained the same result proving Equation 5.34. A similar equation holds for the imaginary parts.

The real part of a product. We might think that an expression similar to Equation 5.34 should hold for a product; that is, multiplication (or division). However, the situation is more complicated. First of all, Equation 5.30a always holds, even for a product:

$$\mathrm{Re}(z_1 z_2) = \frac{1}{2}\left[z_1 z_2 + (z_1 z_2)^\star\right] \tag{5.35}$$

This equation provides a simple way to express the calculation of a product. But we shall show that

$$\mathrm{Re}(z_1 z_2) = 2\,\mathrm{Re}(z_1)\,\mathrm{Re}(z_2) - \mathrm{Re}(z_1^\star z_2) \tag{5.36}$$

which does have some similarity with Equation 5.34; nevertheless, it is definitely different. To prove Equation 5.36, we start by applying Equation 5.33 to Equation 5.35:

$$\mathrm{Re}(z_1 z_2) = \frac{1}{2}\left[z_1 z_2 + (z_1 z_2)^\star\right] = \frac{1}{2}\left[z_1 z_2 + z_1^\star z_2^\star\right]$$

Next, we add zero twice in two different forms:

$$\begin{aligned}&\frac{1}{2}\left[z_1 z_2 + z_1^\star z_2^\star\right] \\ &\quad = \frac{1}{2}\left[z_1 z_2 + (z_1^\star z_2 - z_1^\star z_2) + (z_1 z_2^\star - z_1 z_2^\star) + z_1^\star z_2^\star\right]\end{aligned}$$

Then we rearrange the terms, factor, and use Equations 5.32, 5.33, and 5.30a:

$$\begin{aligned}&\frac{1}{2}\left[z_1 z_2 + (z_1^\star z_2 - z_1^\star z_2) + (z_1 z_2^\star - z_1 z_2^\star) + z_1^\star z_2^\star\right] \\ &\quad = \frac{1}{2}\left[z_1 z_2 + z_1^\star z_2 + z_1 z_2^\star + z_1^\star z_2^\star - z_1^\star z_2 - z_1 z_2^\star\right] \\ &\quad = \frac{1}{2}\left[(z_1 + z_1^\star) z_2 + (z_1 + z_1^\star) z_2^\star - (z_1^\star z_2 + z_1 z_2^\star)\right] \\ &\quad = \frac{1}{2}\left[(z_1 + z_1^\star)(z_2 + z_2^\star) - \left(z_1^\star z_2 + (z_1^\star z_2)^\star\right)\right] \\ &\quad = 2\,\frac{1}{2}(z_1 + z_1^\star)\,\frac{1}{2}(z_2 + z_2^\star) - \frac{1}{2}\left(z_1^\star z_2 + (z_1^\star z_2)^\star\right) \\ &\quad = 2\,\mathrm{Re}(z_1)\,\mathrm{Re}(z_2) - \mathrm{Re}(z_1^\star z_2)\end{aligned}$$

and we have proved Equation 5.36. A somewhat similar equation holds for $\mathrm{Im}(z_1 z_2)$.

It is usually easier to evaluate $\mathrm{Re}(z_1 z_2)$ directly, rather than evaluate the right side of Equation 5.36. We give Equation 5.36 for reference and to show that an equation similar to the one for addition, Equation 5.34, does not exist for multiplication; that is,

$$\mathrm{Re}(z_1 z_2) \neq \mathrm{Re}(z_1)\,\mathrm{Re}(z_2) \tag{5.37}$$

Example 5.2.8 Work with Equations 5.35 and 5.36. (a) Suppose the complex numbers are

$$z_1 = 2 - i3 \qquad \text{and} \qquad z_2 = -5 + i$$

Then we calculate directly

$$z_1 z_2 = (2 - i3)(-5 + i) = -7 + i17$$

and we quickly read the real part as -7. But to show that Equation 5.35 works, we evaluate it:

$$\begin{aligned}\mathrm{Re}(z_1 z_2) &= \frac{1}{2}\left[z_1 z_2 + (z_1 z_2)^\star\right] \\ &= \frac{1}{2}\left[(-7 + i17) + (-7 - i17)\right] = -7\end{aligned}$$

To use Equation 5.36, we first calculate

$$z_1^\star z_2 = (2 + i3)(-5 + i) = -13 - i13$$

and then continue with Equation 5.36. By inspection and calculation, we get

$$2\,\mathrm{Re}(z_1)\,\mathrm{Re}(z_2) - \mathrm{Re}(z_1^\star z_2) = 2(2)(-5) - (-13) = -7$$

which agrees with what we got before.

(b) We use the same complex numbers, but divide:

$$\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right) = (2 - i3)\left(\frac{1}{-5+i}\,\frac{-5-i}{-5-i}\right)$$
$$= (2+i3)\left(-\frac{5}{26} - i\,\frac{1}{26}\right) = -\frac{1}{2} + i\,\frac{1}{2}$$

Then, we can read off the Re part by inspection to get $-1/2$, or use Equation 5.35 to obtain the same result:

$$\mathrm{Re}\left(\frac{z_1}{z_2}\right) = \frac{1}{2}\left[\left(\frac{z_1}{z_2}\right) + \left(\frac{z_1}{z_2}\right)^\star\right]$$
$$= \frac{1}{2}\left[\left(-\frac{1}{2} + i\,\frac{1}{2}\right) + \left(-\frac{1}{2} - i\,\frac{1}{2}\right)\right] = -\frac{1}{2}$$

To apply Equation 5.36, we first calculate

$$z_1^\star\left(\frac{1}{z_2}\right) = (2+i3)\left(-\frac{5}{26} - i\,\frac{1}{26}\right) = -\frac{7}{26} - i\,\frac{17}{26}$$

and then

$$2\,\mathrm{Re}(z_1)\,\mathrm{Re}\left(\frac{1}{z_2}\right) - \mathrm{Re}\left(z_1^\star\frac{1}{z_2}\right)$$
$$= 2(2)\left(-\frac{5}{26}\right) - \left(-\frac{7}{26}\right) = -\frac{1}{2}$$

5.2.10 The basic idea

We frequently deal with functions of the form

$$E = A\cos(bx + \phi) \tag{5.38}$$

which we can write as a complex expression, based on the information in the previous sections (in particular, see Euler's identity as given in Equation 5.11),

$$E = \mathrm{Re}\left[Ae^{i(bx+\phi)}\right] \tag{5.39}$$

We usually want to perform mathematical operations on the complex expression inside the brackets, and it is these mathematical operations that are easier to achieve with the complex expression than the noncomplex one. But it is often awkward to carry the Re notation forward in the mathematical operations when we simply want to manipulate the complex expression. Thus, to remind us that we are working with a complex expression and that in the end we need to take the Re part, we write the name in bold face:

$$\mathbf{E} = Ae^{i(bx+\phi)} \tag{5.40}$$

Some of the quick operations we can perform with this polar notation use a property of exponents:

$$\mathbf{E} = Ae^{i(bx+\phi)} = Ae^{i(bx)}\,e^{i\phi} = A\frac{e^{i(bx)}}{e^{-i\phi}} \tag{5.41}$$

However, the Re part of any of the expressions in Equation 5.41 must equal the equation we started with, namely Equation 5.38. To illustrate how this statement is true, we shall take the last expression in Equation 5.41, apply Euler's identity as listed in Equation 5.11, use some trig identities, multiply, and rearrange:

$$\mathbf{E} = A\,\frac{e^{i(bx)}}{e^{-i\phi}} = A\,\frac{\cos bx + i\sin bx}{\cos\phi - i\sin\phi}$$
$$= A\,\frac{\cos bx + i\sin bx}{\cos\phi - i\sin\phi}\,\frac{\cos\phi + i\sin\phi}{\cos\phi + i\sin\phi}$$
$$= A\left[\frac{\cos bx\,\cos\phi - \sin bx\,\cos\phi}{\cos^2\phi + \sin^2\phi}\right.$$
$$\left. + i\,\frac{\sin bx\,\cos\phi + \cos bx\,\sin\phi}{\cos^2\phi + \sin^2\phi}\right]$$
$$= A\cos(bx+\phi) + i\,A\sin(bx+\phi)$$

and we get the desired result of

$$E = \mathrm{Re}(\mathbf{E}) = A\cos(bx+\phi)$$

which agrees with Equation 5.38. But, we had to work hard to show this agreement; the quick and efficient ways of manipulating harmonic expressions as shown in Equation 5.41 is the reason for using complex notation.

However, we must be careful. Suppose we want to square E in Equation 5.38; that is,

$$E^2 = A^2\cos^2(bx+\phi) \tag{5.42}$$

Like before, we set

$$\mathbf{E} = Ae^{i(bx+\phi)} \tag{5.43}$$

so that

$$E = \mathrm{Re}(\mathbf{E}) = A\cos(bx+\phi)$$

But then

$$E^2 = E\,E = \mathrm{Re}(\mathbf{E})\,\mathrm{Re}(\mathbf{E})$$

which is not equal to $\mathrm{Re}(\mathbf{E}^2)$, or $\mathrm{Re}(\mathbf{E}\,\mathbf{E})$, according to Equations 5.36 and 5.37 (the boldface $\mathbf{E}$s correspond to the zs).

A cleaner method for handling this situation is to use Equation 5.30a and write

$$E = \text{Re}(\mathbf{E}) = \frac{1}{2}\left(\mathbf{E} + \mathbf{E}^{\star}\right) \tag{5.44}$$

Then

$$\begin{aligned} E^2 &= (\text{Re}(\mathbf{E}))^2 \\ &= \frac{1}{4}\left(\mathbf{E} + \mathbf{E}^{\star}\right)^2 \\ &= \frac{1}{4}\left[\mathbf{E}^2 + \left(\mathbf{E}^{\star}\right)^2 + 2\mathbf{E}\mathbf{E}^{\star}\right] \end{aligned} \tag{5.45}$$

As a check, to show that this equation yields Equation 5.42, we substitute Equation 5.43, use Euler's identity, use a trig identity, and manipulate:

$$\begin{aligned} E^2 &= \frac{1}{4}\left[\mathbf{E}^2 + \left(\mathbf{E}^{\star}\right)^2 + 2\mathbf{E}\mathbf{E}^{\star}\right] \\ &= \frac{1}{4}\left[\left(Ae^{i(bx+\phi)}\right)^2 + \left(Ae^{-i(bx+\phi)}\right)^2 + 2A^2\right] \\ &= \frac{A^2}{4}\left[e^{i2(bx+\phi)} + e^{-i2(bx+\phi)} + 2\right] \\ &= \frac{A^2}{2}\left[\cos 2(bx+\phi) + 1\right] \\ &= \frac{A^2}{2}\left[2\cos^2(bx+\phi)\right] \\ &= A^2\cos^2(bx+\phi) \end{aligned}$$

which does agree with Equation 5.42.

5.3 Harmonic Waves

5.3.1 Traveling harmonic waves

We start with a very simple harmonic wave in space (that is, in a vacuum):

$$E = E_0 \cos \frac{2\pi}{\lambda} z \tag{5.46}$$

which we graph in Figure 5.15. The variable E is called the

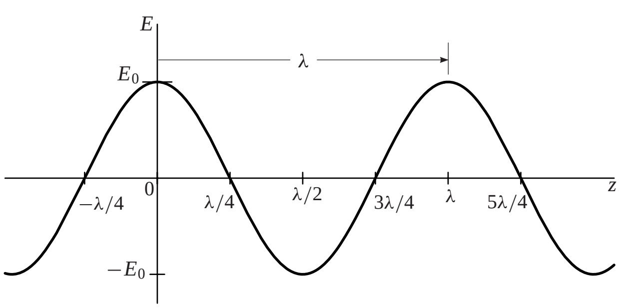

Figure 5.15

displacement, the coefficient E_0 the **amplitude**, and λ the **spatial period**, or **wavelength** (the distance in one cycle, or one repetitive unit).

Travel in the $+z$ direction. Now we imagine that the wave moves to the right with the speed v, so that in Figure 5.16 the solid curve represents the wave position at time $t = 0$, and the dashed curve the wave position at a later time $t = t$. If we imagine a new coordinate system, the z' coordinate system,

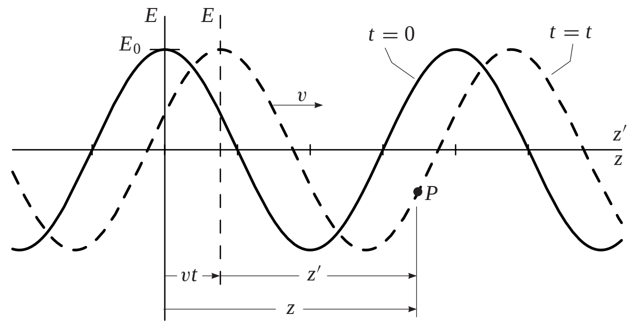

Figure 5.16

to move with the dashed wave, then we see that any point P on this wave is described by an equation in terms of z' that is just like the one in Equation 5.46, namely

$$E = E_0 \cos \frac{2\pi}{\lambda} z' \tag{5.47}$$

As we show in Figure 5.16, in a time t the z' origin (the point where the vertical dashed line crosses the z, z' axis) moves a distance vt, and we read off for any point P on the moving wave

$$z' = z - vt \tag{5.48}$$

Substituting Equation 5.48 in Equation 5.47, we obtain an equation that describes the traveling wave in terms of z, as well as t,

$$E = E_0 \cos \frac{2\pi}{\lambda}(z - vt) \tag{5.49}$$

which is an equation of a traveling harmonic wave down the positive z axis. We observe that the traveling harmonic wave in Equation 5.49 is not only a function of z and t, but also of $(z - vt)$; in fact, all waves that travel along the positive z axis, whether harmonic or not, are functions of $(z - vt)$.

We use the cosine function to describe our harmonic waves simply for convenience: first of all, the cosine function is an even function; and second, the real part of a complex expression in polar form involves the cosine. However, the description of harmonic waves could be done in terms of the sine function just as well.

Light is an electromagnetic field composed of electric and magnetic fields that travel together through a vacuum or other media. Because the human eye is sensitive only to the electric field, we use E and E_0 to represent the electric field in the preceding equations. In terms of SI units, a convenient unit of electric field is the volt/meter, or V/m; for z, λ the meter, and for t the second. However, multiples of these units may be preferable in some problems.

Travel in the $-z$ direction. Now suppose the harmonic wave travels down the negative z axis, instead of the positive z. Then the diagram that is analogous to Figure 5.16 is shown in Figure 5.17. We insert a minus sign in front of vt because

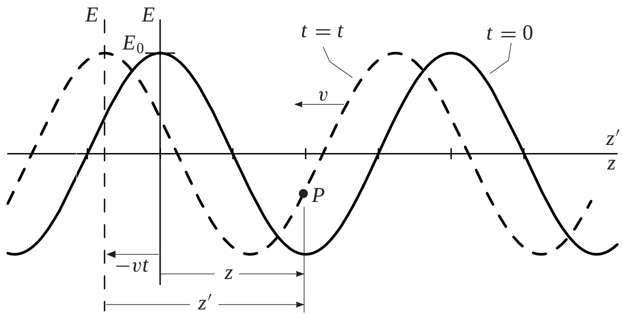

Figure 5.17

both the speed v and time t are positive quantities, and the vt direction is in the negative direction, as we see from Figure 5.17. Just as before in Figure 5.16, the dashed E axis moves with the traveling dashed wave, and the equation for this wave is the same as Equation 5.47 (we renumber for convenience):

$$E = E_0 \cos \frac{2\pi}{\lambda} z' \tag{5.50}$$

Treating the dimension arrows for z', z, and $-vt$ like vectors, as we have done in previous chapters, we read off by inspection of Figure 5.17

$$z' = z - (-vt) = z + vt \tag{5.51}$$

We substitute Equation 5.51 into Equation 5.50, and obtain the equation of a harmonic wave traveling along the negative z axis:

$$E = E_0 \cos \frac{2\pi}{\lambda} (z + vt) \tag{5.52}$$

We observe the interesting feature of these traveling wave equations: those that represent travel along the positive z axis are functions of $(z - vt)$; those along the negative z axis are functions of $(z + vt)$.

The phase θ and the phase constant ϕ. We do one more thing to traveling harmonic waves to make them more general: we subtract a constant ϕ (in radians) so that for a harmonic wave traveling in the $+z$ direction

$$E = E_0 \cos \frac{2\pi}{\lambda} (z - vt) \tag{5.53}$$

it becomes

$$E = E_0 \cos \left[\frac{2\pi}{\lambda} (z - vt) - \phi \right] \tag{5.54}$$

We subtract ϕ because it makes its effect somewhat easier to describe; however, ϕ can be positive or negative (or zero). The argument of the cosine function is called the **phase** θ; that is, in equation form

$$\theta = \text{phase} = \frac{2\pi}{\lambda} (z - vt) - \phi \tag{5.55}$$

where ϕ is the initial phase or the **phase constant**. To see the effect of a positive phase constant, we look at waves when $t = 0$. Then, when ϕ is zero, Equation 5.54 becomes

$$E = E_0 \cos \frac{2\pi}{\lambda} z \tag{5.56}$$

as shown by the solid curve in Figure 5.18. The wave peak at the origin occurs when the phase θ is zero; thus

$$\theta = \frac{2\pi}{\lambda} z_1 = 0 \qquad \text{or} \qquad z_1 = 0 \tag{5.57}$$

Now we allow ϕ to have a positive value, and continue to let $t = 0$. Equation 5.54 reads in this case

$$E = E_0 \cos \left(\frac{2\pi}{\lambda} z - \phi \right) \tag{5.58}$$

The peak that was at the origin is still described by $\theta = 0$, so we have

$$\theta = \frac{2\pi}{\lambda} z_2 - \phi = 0 \qquad \text{or} \qquad z_2 = \frac{\lambda}{2\pi} \phi \tag{5.59}$$

From Equations 5.57 and 5.59, we find

$$\Delta z = z_2 - z_1 = \frac{\lambda}{2\pi} \phi \tag{5.60}$$

that is, the entire harmonic wave is shifted to the right by the amount Δz, as shown by the dashed curve in Figure 5.18. We also see that if we set $z = 0$ in Equation 5.58, we have

$$E = E_0 \cos(-\phi) = E_0 \cos \phi \tag{5.61}$$

where the last step follows because the cosine is an even function. The value given by $E_0 \cos \phi$ is the point where the dashed curve crosses the E axis, as indicated in Figure 5.18.

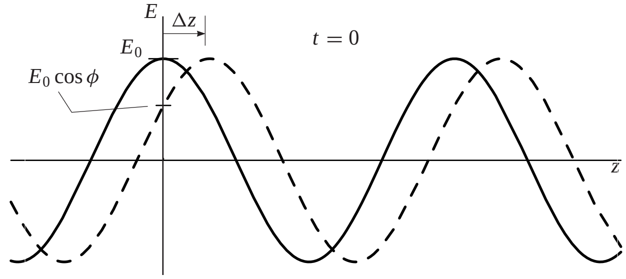

Figure 5.18

Thus, the effect of the phase constant ϕ is to shift the entire harmonic wave along the z axis. Summarizing, for fixed t, a positive ϕ shifts the wave in the positive z direction; a negative ϕ shifts the wave in the negative z direction.

It is important to note that to follow the movement of the wave peak that was at the origin (look at the solid curve in Figure 5.18), we set the phase $\theta = 0$—see Equations 5.57 and 5.59. By choosing other values for z and t, we could obtain another value for θ, and thereby follow the movement of another point on the wave.

The period T. So far we have graphed a traveling harmonic wave in terms of z at particular values of t. Now we want to do the opposite: graph the wave in terms of t at particular values of z. First, we imagine a harmonic wave traveling in the $+z$ direction with a speed v: in Figure 5.19, we show the waves at the times of $t = 0$, t_1, and t_2 to help with the visualization. To obtain a graph of the wave as a function of t at a particular value of z, we choose a plane at $z = z_1 = \lambda$. Then, we imagine that we set up a device that measures the E value of the wave with time t as it passes the plane at $z = z_1 = \lambda$, as indicated in Figure 5.19—we show the E values we measure at times $t = 0$, t_1, and t_2 by the points 0, 1, 2. The graph we obtain as a function of time is then shown in Figure 5.20. In this diagram, we denote the time period (the time in one cycle), or simply the **period**, by the symbol T. The period T plays the same role in time as the wavelength λ does in space (see Figure 5.15).

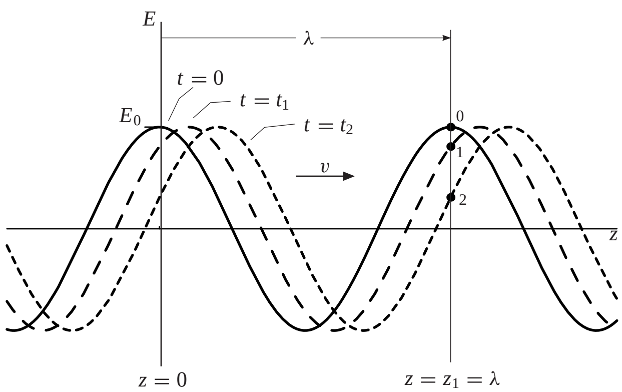

Figure 5.19

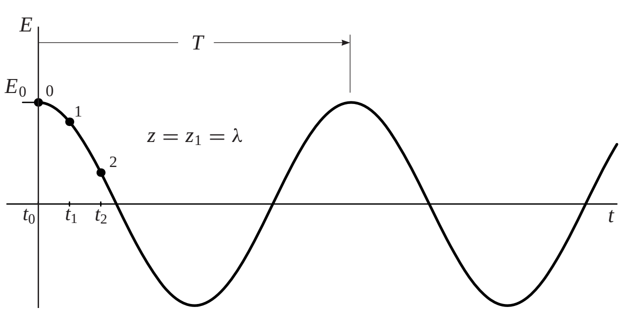

Figure 5.20

If we choose a plane somewhat to the right of $z = z_1 = \lambda$ in Figure 5.19, say at $z = z_2$, then we obtain another graph of E versus t, shown by the dashed curve in Figure 5.21.

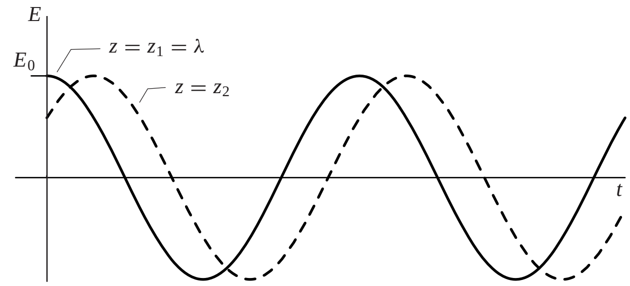

Figure 5.21

Now we return to Figure 5.19, and concentrate on the solid curve. We imagine it to move to the right, and consider how long it will take for that curve to move through the distance of λ. We quickly see that the time would be just the period T that is shown in Figure 5.20. Recalling that $time = distance/speed$, we have

$$T = \frac{\lambda}{v} \tag{5.62}$$

With Equation 5.62, we can rewrite the traveling harmonic wave equation of Equation 5.54 as follows:

$$\begin{aligned} E &= E_0 \cos\left[\frac{2\pi}{\lambda}(z - vt) - \phi\right] \\ &= E_0 \cos\left[\frac{2\pi}{\lambda}\left(z - \frac{\lambda}{T}t\right) - \phi\right] \\ &= E_0 \cos\left[2\pi\left(\frac{z}{\lambda} - \frac{t}{T}\right) - \phi\right] \end{aligned} \tag{5.63}$$

This form of the harmonic wave equation emphasizes the similar roles that λ and T have relative to space and time.

The frequency ν and the wave number σ. To help understand frequency, we look at Figure 5.20, and observe

$$\text{period} = T\ \frac{\text{s}}{\text{cycle}}$$

The **frequency** ν is defined as the reciprocal of the period T; thus,

$$\nu = \text{frequency} = \frac{1}{T}\ \frac{\text{cycles}}{\text{s}} \tag{5.64}$$

The units, cycles/s, indicate that frequency refers to the number of cycles in a second. The cycle unit is a tag unit; that is, it is not a real unit like the meter or the second. A tag unit is used to help understanding, and can be erased whenever it is not needed; for that reason, the frequency unit is written simply as s^{-1}, or the SI unit is used: hertz (Hz).

In space, the analog of the frequency ν is called the linear wave number, or more simply the wave number σ. Following the format we just used to define frequency, we look at Figure 5.15, and observe that

$$\text{spatial period} = \text{wavelength} = \lambda \ \frac{\text{m}}{\text{cycle}}$$

The **wave number** σ is defined as the reciprocal of the wavelength, so that

$$\sigma = \text{wave number} = \frac{1}{\lambda}\ \frac{\text{cycles}}{\text{m}} \qquad (5.65)$$

Here, the units of cycles/m are interpreted as the number of cycles in a meter. Again, the cycle is a tag unit, and the unit for σ is often written as m^{-1} and read as reciprocal meters. No special name is assigned to this unit.

With the definitions in Equations 5.64 and 5.65, we can rewrite Equation 5.63 for a traveling harmonic wave as

$$E = E_0 \cos\left[2\pi\left(\frac{z}{\lambda} - \frac{t}{T}\right) - \phi\right]$$

$$= E_0 \cos\left[2\pi\,(\sigma z - \nu t) - \phi\right] \qquad (5.66)$$

We make two more definitions to simplify Equation 5.66.

The angular wave number k and angular frequency ω. The final set of definitions we make streamline the traveling harmonic wave equation yet more. We define the **angular wave number** k as

$$k = \frac{2\pi}{\lambda} = 2\pi\sigma \qquad (5.67)$$

and the **angular frequency** ω as

$$\omega = \frac{2\pi}{T} = 2\pi\nu \qquad (5.68)$$

Multiplying the 2π into the parentheses in Equation 5.66, and then substituting Equations 5.67 and 5.68, we get

$$E = E_0 \cos\left(kz - \omega t - \phi\right) \qquad (5.69)$$

which is the simplest form of the traveling harmonic wave equation; it is the one usually used in theoretical work. The units for k are the rad/m and for ω the rad/s.

5.3.2 Representation in the complex plane

As a complex expression in polar form, we write Equation 5.69 as

$$\mathbf{E} = E_0\, e^{i(kz - \omega t - \phi)} \qquad (5.70)$$

where we write $\mathbf{E}$ in bold face to remind us that we have a complex expression from which we must take the Re part to obtain Equation 5.69, as discussed in Section 5.2.10.

Next, we want to show how the phasor representation of Equation 5.70 can be interpreted graphically to give the harmonic wave E as a function of z at a fixed value of t, say at $t = t_0$. The result is shown in Figure 5.22; we now explain how this diagram is constructed.

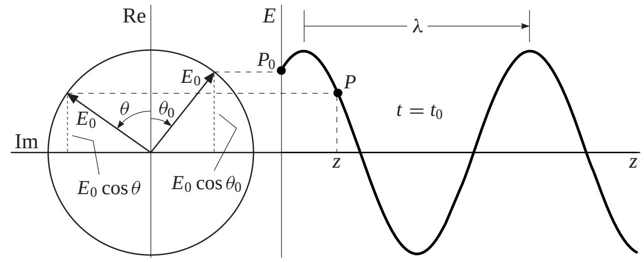

Figure 5.22

In terms of the phase θ, Equation 5.70 becomes

$$\mathbf{E} = E_0\, e^{i\theta} \qquad (5.71)$$

where

$$\theta = kz - \omega t - \phi \qquad (5.72)$$

At the particular time $t = t_0$, we write Equation 5.72 as

$$\theta = kz - \omega t_0 - \phi = kz + \theta_0 \qquad (5.73)$$

where

$$\theta_0 = -\omega t_0 - \phi = -(\omega t_0 + \phi) \qquad (5.74)$$

Equation 5.71 indicates that the phasor magnitude is always E_0, so we draw a circle of this radius in Figure 5.22. To make it easier to represent the cosine, we rotate the complex plane 90 deg counterclockwise to make the Re axis vertical. At $z = 0$, Equation 5.73 gives $\theta = \theta_0$, where by Equation 5.74, $\theta_0 < 0$ for $\omega t_0 + \phi > 0$. Thus, we draw θ_0 as a negative angle in Figure 5.22. By inspection, we see that the projection of this phasor on the Re axis is $E_0 \cos\theta_0$, as noted in the diagram. Continuing to look at the diagram, a dashed line is drawn from the head of this phasor to the P_0 point on the E versus z graph, the beginning point on this graph at $z = 0$. Now let z increase from 0, so that θ also increases according to Equation 5.73 making the phasor rotate counterclockwise in the complex plane. We show one of the later locations of this phasor in Figure 5.22 with its vertical projection on the Re axis of $E_0 \cos\theta$; the dashed line from the head of this phasor locates the point P on the harmonic wave graph. In this manner, the rotating phasor builds up the E versus z graph in Figure 5.22. Note how the phasor making a complete rotation of 2π rad moves the point P through the wavelength λ (the spatial period) on the wave.

Representing a harmonic wave in polar form (see Equation 5.70) in the complex plane is a helpful way to visualize the wave. In Figure 5.22, we represented the case for z varies with a fixed time t. A similar procedure can be described to get the E versus t graph at a fixed value of z.

5.3.3 Summary of traveling harmonic waves

A traveling harmonic wave has a very simple form when graphed, but it is more difficult to describe analytically because of the several different ways to do it, as we have seen in the previous sections. In its simplest form, we write

$$E = E_0 \cos(kz - \omega t - \phi) \tag{5.75}$$

which is the real part of the complex expression

$$\mathbf{E} = E_0 \, e^{i(kz - \omega t - \phi)} \tag{5.76}$$

In Section 5.3.1, we learned that k and ω could be expressed in terms of other quantities, each meaningful in its own way in a physical sense. To try to summarize these different ways of viewing a traveling harmonic wave, we might say it is a single entity like a coin with two sides: one side a space view, the other side a time view—each view with its own set of parameters. We summarize these two views in the table of Figure 5.23. In the last row of this table, we write the speed v of wave travel including several forms of what v equals, starting with Equation 5.62, which we have solved for v. We note that each of the expressions for v contain two symbols, one from the space column, the other from the time column: in this manner, the speed v combines the two views of the traveling harmonic wave.

space	time
wavelength λ	period T
wave number $\sigma = \dfrac{1}{\lambda}$	frequency $\nu = \dfrac{1}{T}$
angular wave number $k = \dfrac{2\pi}{\lambda} = 2\pi\sigma$	angular frequency $\omega = \dfrac{2\pi}{T} = 2\pi\nu$
speed $v = \dfrac{\lambda}{T} = \lambda\nu = \dfrac{1}{\sigma T} = \dfrac{\omega}{k}$	

Figure 5.23 The parameters of a traveling harmonic wave.

As we have mentioned before, Equation 5.75 (or Equation 5.76) describes a harmonic wave traveling in the $+z$ direction. This wave represents a light wave in terms of the electric field E (there is also a magnetic field H that we ignore here for simplicity, since the eye only responds to the electric field), and it is traveling in the simplest medium that exists, the space of a vacuum. To represent position in this medium, we use an x, y, z coordinate system: then, at every x, y, z point in this system we associate an E value given by Equation 5.75. Thus, we imagine harmonic waves traveling in the $+z$ direction everywhere in this space. To give a picture, we choose a particular value of t, say $t = t_1$, and draw the diagram shown in Figure 5.24. In this way, we make a snapshot of a portion of the harmonic waves traveling in space. We also show two surfaces called wavefronts, which are surfaces of constant phase θ. For example, when $z = z_1$, we look at Equation 5.75 (or Equation 5.76), and see that on this surface

$$\theta = \theta_1 = kz_1 - \omega t_1 - \phi$$

and on the surface $z = z_2$, the phase θ has another constant value. In general, any surface of constant phase is called a **wavefront**. Since in this particular case, the surfaces are planes, harmonic waves of this type are called **plane waves**, and represent the simplest of light waves. In this special case, we note that the E value is also constant on these surfaces, but in general, that is not true.

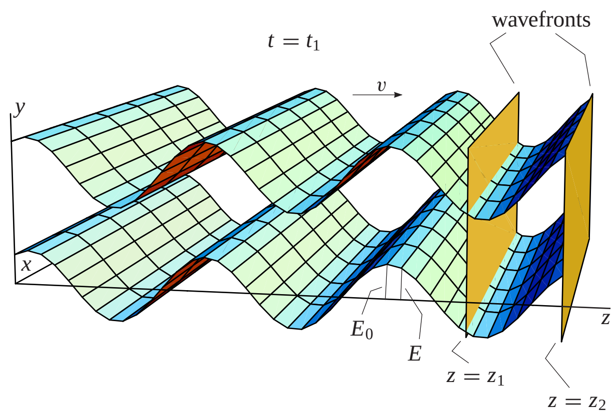

Figure 5.24 Plane waves.

We note that all the E values in Figure 5.24 are perpendicular to the z axis, the direction of travel; that is, light waves are transverse waves. Although plane waves are simple waves, they are very important because complicated waves can be constructed by adding waves of this type with the methods of Fourier series and integrals.

5.3.4 The intensity I

All traveling waves carry energy in the direction of travel. Sound waves have this property; electromagnetic waves also have this property. Visible light waves are electromagnetic waves having wavelengths in a vacuum of 400 to 700 nm (see Figure 0.1); therefore, visible light waves carry energy in the direction of travel. Solving the equation $v = \lambda\nu$ in the table of Figure 5.23 for ν, we calculate,

$$\nu = \frac{v}{\lambda} = \frac{3 \times 10^8 \text{ m/s}}{\begin{Bmatrix} 400 \text{ nm} \\ 700 \text{ nm} \end{Bmatrix}} = \begin{cases} 750 \times 10^{12} \text{ Hz} \\ 430 \times 10^{12} \text{ Hz} \end{cases} \tag{5.77}$$

where $v = c = 3 \times 10^8$ m/s for the speed of light in a vacuum. We note the high frequencies of visible light waves.

We represent the energy carried by the light waves with the symbol U. Now suppose we have a detector of small surface area A that is able to measure U, and is placed in the path of the light waves oriented perpendicular to the direction

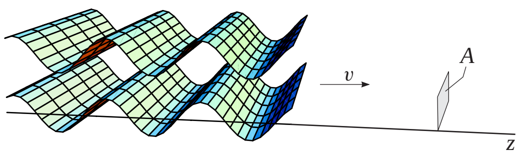

Figure 5.25

of wave travel, as shown in Figure 5.25. If the detector measures an energy dU passing through A in a time dt, then we define the **instantaneous intensity** I_{inst} as

$$I_{\text{inst}} = \frac{1}{A}\frac{dU}{dt} \tag{5.78}$$

where the SI units for I_{inst} are joule/second $\cdot$ meter^2 or, the equivalent form, watt/meter^2; in symbol form: $\text{J/s}\cdot\text{m}^2$ or W/m^2. In electromagnetic theory, it can be shown that Equation 5.78 leads to

$$I_{\text{inst}} = C\,E^2 \tag{5.79}$$

where C is a constant with SI units of watt/volt^2, or W/V^2. The value of this constant is not needed in our work. Equation 5.75 gives E for a traveling harmonic wave, which we rewrite for convenience:

$$E = E_0 \cos(kz - \omega t - \phi) \tag{5.80}$$

Substituting Equation 5.80 into Equation 5.79 gives the instantaneous intensity for a traveling harmonic wave, namely

$$I_{\text{inst}} = C\,E_0^2 \cos^2(kz - \omega t - \phi) \tag{5.81}$$

Because $\cos(-\theta) = \cos\theta$, and using Equation 5.68 to write $\omega = 2\pi/T$, we express Equation 5.81 in a better form for our next task, determining the time average of I_{inst}:

$$\begin{aligned} I_{\text{inst}} &= C\,E_0^2 \cos^2(kz - \omega t - \phi) \\ &= C\,E_0^2 \cos^2(-kz + \omega t + \phi) \\ &= C\,E_0^2 \cos^2\left(\frac{2\pi}{T}t + \alpha\right) \end{aligned} \tag{5.82}$$

where

$$\alpha = -kz + \phi \tag{5.83}$$

is independent of the time t. We now observe that the high frequencies for visible light given in Equation 5.77 make the period $T = 1/\nu$ (see Equation 5.64) range from 1.33 fs to 2.33 fs, where fs = femtosecond = 10^{-15} s. These extremely small periods make the time variation of I_{inst} so rapid that practical detectors (such as the eye, photographic film, photocell, and all other detectors that we can construct) do not respond to I_{inst}. Instead, these detectors respond to the time average (or mean) with respect to time of I_{inst}, which we shall simply call the intensity I.

In calculus, the average or mean of a function $f(t)$ over an interval from t_1 to t_2 is defined to be

$$f_{\text{av}} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(t)\,dt \tag{5.84}$$

We use this definition to calculate the time average of I_{inst} over a time interval from $t_1 = 0$ to $t_2 = \tau$ with the time $\tau >> T$ to correspond to practical detectors. Then, using the trig identity $2\cos^2 A = 1 + \cos 2A$, and substituting Equation 5.82 for I_{inst}, we have

$$\begin{aligned} I = (I_{\text{inst}})_{\text{av}} &= \frac{1}{\tau}\int_0^{\tau>>T} I_{\text{inst}}\,dt \\ &= \frac{1}{\tau}\int_0^{\tau} C\,E_0^2 \cos^2\left(\frac{2\pi}{T}t + \alpha\right) dt \\ &= \frac{C\,E_0^2}{2\tau}\int_0^{\tau} dt + \frac{C\,E_0^2}{2\tau}\int_0^{\tau}\cos\left(\frac{4\pi}{T}t + 2\alpha\right) dt \\ &= \frac{C\,E_0^2}{2} + \frac{T}{\tau}\frac{C\,E_0^2}{8\pi}\left[\sin\left(\frac{4\pi}{T}\tau + 2\alpha\right) - \sin 2\alpha\right] \\ &\approx \frac{C}{2}E_0^2 \\ &= C\,E_0^2 = C\,\mathbf{E}\,\mathbf{E}^\star \end{aligned} \tag{5.85}$$

where we have used $\tau >> T$ to remove the expression involving the sin terms within the brackets, replaced $C/2$ by C for simplicity (both $C/2$ and C are constants whose values we never need, so it is a type of shorthand to use the same C for both constants), and in the last step we represent the E of Equation 5.80 in its complex form, namely,

$$\mathbf{E} = E_0\, e^{i(kz - \omega t - \phi)} \tag{5.86}$$

The $I = C\,\mathbf{E}\,\mathbf{E}^\star$ expression in Equation 5.85 provides a very advantageous way of quickly calculating the **time average intensity** I, or for short, the **intensity** I. To see how $\mathbf{E}\,\mathbf{E}^\star$ quickly gives the amplitude squared E_0^2, you might want to review Section 5.2.7 and Equation 5.29, where $|z|$ represents (or equals) E_0. Even when $\mathbf{E}$ is not given simply by Equation 5.86, but is a sum of several harmonic waves with the same frequency (but with the other parameters different), the calculation of $\mathbf{E}\,\mathbf{E}^\star$ gives the correct square of the amplitude. The SI units for I are the same as for I_{inst}, namely, $\text{J/s}\cdot\text{m}^2$ or W/m^2.

5.3.5 Traveling harmonic waves in other media

So far, we have assumed that the harmonic waves travel with speed c in a vacuum, a homogeneous and isotropic medium of index of refraction $n = 1$. The space and time parameters (see Figure 5.23) of a traveling harmonic wave in a vacuum are written without a subscript. Now suppose a harmonic wave enters another homogeneous, isotropic medium of index of refraction $n > 1$; then the speed of wave travel slows to $v < c$. Electromagnetic theory shows that the time parameters do not change; that is, T, ν, and ω remain the same. With definition of n, as given in Equation 1.1, and with the help of the relationships in Figure 5.23, we obtain

$$n = \frac{\text{speed in a vacuum}}{\text{speed in medium}} = \frac{c}{v} = \frac{\lambda\nu}{\lambda_n\nu} = \frac{\lambda}{\lambda_n} \qquad (5.87)$$

which says that

$$\lambda_n = \frac{\lambda}{n} \qquad (5.88)$$

Thus, the wavelength shortens when the harmonic wave travels into a medium with a greater index of refraction, as we illustrate in Figure 5.26, where the interface is the surface of the medium, for example, glass. The other space parameters of σ_n and k_n in the medium would also have different values from those in the vacuum.

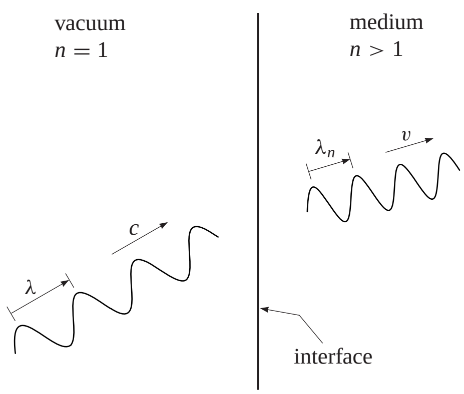

Figure 5.26

We now want to look in greater detail at harmonic waves that travel through one or more different media.

5.3.6 The importance of the optical path difference Δ and the phase difference δ

Suppose that two harmonic waves of the same angular frequency ω travel in a homogeneous, isotropic medium of index of refraction n. One wave travels along the z_1 axis, the other along the z_2 axis, as shown in Figure 5.27. Based on the description of harmonic waves that we gave in previous sections, we write the equations of these waves as

$$E_1 = E_{01} \cos(k_n z_1 - \omega t - \phi_1) \qquad (5.89a)$$

$$E_2 = E_{02} \cos(k_n z_2 - \omega t - \phi_2) \qquad (5.89b)$$

where the amplitudes E_{01}, E_{02} are not necessarily the same, and where Figure 5.23 and Equation 5.88 gives

$$k_n = \frac{2\pi}{\lambda_n} = \frac{2\pi}{\lambda} n \qquad (5.90)$$

The origin O_1 marks the point where $z_1 = 0$, and O_2 is the origin for the z_2 axis, where $z_2 = 0$. The respective phases of E_1 and E_2 are

$$\theta_1 = k_n z_1 - \omega t - \phi_1 \quad \text{and} \quad \theta_2 = k_n z_2 - \omega t - \phi_2$$

At the origins, z_1 and z_2 equal zero, so then the respective phases at the origins become

$$\theta_1 = -\omega t - \phi_1 \quad \text{and} \quad \theta_2 = -\omega t - \phi_2 \qquad (5.91)$$

We wish to make these waves in phase at their respective origins; that is, if we imagine that we could see these waves at O_1 and O_2, then we would see them both going through

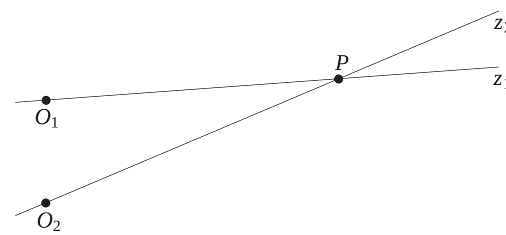

Figure 5.27 Two harmonic waves traveling in a homogeneous, isotropic medium of index of refraction n.

their maxima, minima, and zeros at the same time—that is, the respective phase equations must be the same. Looking at Equations 5.91, we see that we satisfy this requirement by making $\phi_1 = \phi_2 = \phi$. Then we have

$$\theta_1 = -\omega t - \phi \quad \text{and} \quad \theta_2 = -\omega t - \phi \qquad (5.92)$$

Now we can say that E_1 and E_2 are in phase with each other at O_1 and O_2; that is

$$E_1 = E_{01} \cos(-\omega t - \phi) \qquad (5.93a)$$

$$E_2 = E_{02} \cos(-\omega t - \phi) \qquad (5.93b)$$

at their respective origins. Because the cosine function is an even function, we could remove the minus signs in Equations 5.93, but for what we want to do next, it is simpler to leave them in place. With the phase constants the same, Equations 5.89 become

$$E_1 = E_{01} \cos(k_n z_1 - \omega t - \phi) \qquad (5.94a)$$

and

$$E_2 = E_{02} \cos(k_n z_2 - \omega t - \phi) \qquad (5.94b)$$

Inspecting Figure 5.27, we see that the axes cross at the point P; at this point our harmonic waves in Equations 5.94 become

$$E_1 = E_{01}\cos(k_n z_{1P} - \omega t - \phi) \tag{5.95a}$$

$$E_2 = E_{02}\cos(k_n z_{2P} - \omega t - \phi) \tag{5.95b}$$

where z_{1P} is the distance from O_1 to P, and in like manner, z_{2P} is the distance from O_2 to P.

By inspection of Equations 5.95, we see that the phases of the two harmonic waves at the point P are

$$\theta_{1P} = k_n z_{1P} - \omega t - \phi \tag{5.96a}$$

$$\theta_{2P} = k_n z_{2P} - \omega t - \phi \tag{5.96b}$$

The **phase difference** δ is defined as the difference between these phases, so with Equation 5.90 we get

$$\begin{aligned}\delta &= \theta_{2P} - \theta_{1P} \\ &= (k_n z_{2P} - \omega t - \phi) - (k_n z_{1P} - \omega t - \phi) \\ &= k_n (z_{2P} - z_{1P}) = \frac{2\pi}{\lambda_n}(z_{2P} - z_{1P}) \\ &= \frac{2\pi}{(\lambda/n)}(z_{2P} - z_{1P}) = \frac{2\pi}{\lambda}(nz_{2P} - nz_{1P}) \\ &= \frac{2\pi}{\lambda}(\ell_{2P} - \ell_{1P}) = \frac{2\pi}{\lambda}\Delta \end{aligned} \tag{5.97}$$

where

$$\Delta = \ell_{2P} - \ell_{1P} = nz_{2P} - nz_{1P} \tag{5.98}$$

is called the **optical path difference**. We frequently want to know the phase difference δ when we know the optical path difference Δ, so Equations 5.97 and 5.98 are important equations. The distance $\ell_{1P} = nz_{1P}$ is called the **optical path** (or the optical path length) from O_1 to P, and $\ell_{2P} = nz_{2P}$ is given a similar explanation. The distance z_{1P} is called the **geometrical path** (or the geometrical path length) from O_1 to P; similarly for z_{2P}.

The phase difference δ helps to simplify work with harmonic waves of the type shown in Equations 5.95, which we emphasize, describe the behavior of the harmonic waves with time t at the point P. If we imagine an observer at the point P to graph these harmonic waves as functions of time t, the graphs obtained might look like those in Figure 5.28.

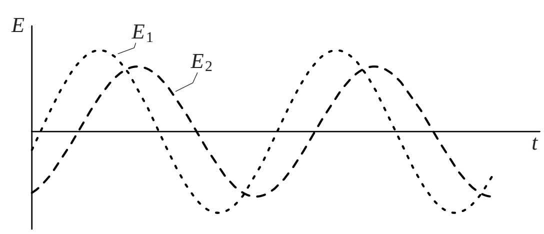

Figure 5.28

Because we are looking at these waves as functions of time t, it helps if we write the arguments (the phases) in Equations 5.95 with the ωt first. We remember that the cosine function is an even function; that is, $\cos(-\theta) = \cos\theta$. Therefore, we multiply the arguments through by -1, rearrange slightly, and obtain the equivalent equations

$$E_1 = E_{01}\cos(\omega t - k_n z_{1P} + \phi) \tag{5.99a}$$

$$E_2 = E_{02}\cos(\omega t - k_n z_{2P} + \phi) \tag{5.99b}$$

We graph these equations again in Figure 5.29, but also indicate that we shift the time origin by a time of t_0 to a new t' axis origin at the E_1 peak so that we have

$$E_1 = E_{01}\cos(\omega t') \tag{5.100}$$

which is a much simpler expression than the one in Equation 5.99a. To determine t_0, we start by observing in the

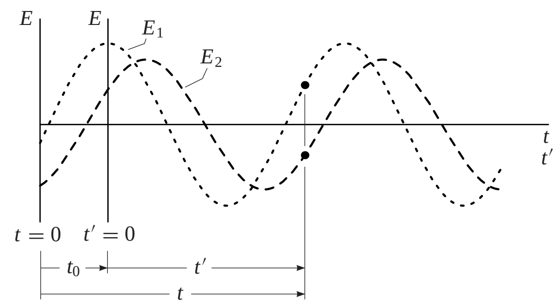

Figure 5.29

diagram of Figure 5.29 that arbitrary points on the waves are described in terms of either t or t' where

$$t = t' + t_0 \tag{5.101}$$

We then substitute Equation 5.101 into Equation 5.99a, and set it equal to the desired Equation 5.100 to get

$$E_{01}\cos[\omega(t' + t_0) - k_n z_{1P} + \phi] = E_{01}\cos(\omega t')$$

$$\omega(t' + t_0) - k_n z_{1P} + \phi = \omega t'$$

which when solved for t_0 gives

$$t_0 = \frac{k_n z_{1P} - \phi}{\omega} \tag{5.102}$$

Next we substitute Equation 5.101 into Equation 5.99b, replace the t_0 by Equation 5.102, and rearrange:

$$\begin{aligned} E_2 &= E_{02}\cos[\omega(t' + t_0) - k_n z_{2P} + \phi] \\ &= E_{02}\cos[\omega t' - k_n(z_{2P} - z_{1P})] \end{aligned}$$

Substituting Equation 5.90 for k_n, and then with the help of Equations 5.98 and 5.97, we finally obtain

$$\begin{aligned} E_2 &= E_{02}\cos\left[\omega t' - \frac{2\pi}{\lambda}(nz_{2P} - nz_{1P})\right] \\ &= E_{02}\cos\left[\omega t' - \frac{2\pi}{\lambda}\Delta\right] \\ &= E_{02}\cos(\omega t' - \delta) \end{aligned} \tag{5.103}$$

We have now achieved an important result. Whenever we want to describe two harmonic waves of the form in Equations 5.95 or 5.99, we can write them very simply in terms of the phase difference δ as

$$E_1 = E_{01}\cos(\omega t') \tag{5.104a}$$

$$E_2 = E_{02}\cos(\omega t' - \delta) \tag{5.104b}$$

Because there is no t_0 in the equations, and because we normally do not need to know it, we drop the prime and write the equations yet more simply as

$$E_1 = E_{01}\cos(\omega t) \tag{5.105a}$$

$$E_2 = E_{02}\cos(\omega t - \delta) \tag{5.105b}$$

or in the complex plane as

$$E_1 = E_{01}e^{i(\omega t)} \tag{5.106a}$$

$$E_2 = E_{02}e^{i(\omega t - \delta)} \tag{5.106b}$$

where δ has all the information we need to state how the two harmonic waves differ from each other. Even when harmonic waves travel through media that are more complicated than those shown in Figure 5.27, the above equations still work: for example, the waves might travel through several different media on the way to P, or one wave might travel through one set of media, and the other through another set.

The optical path has an interesting interpretation that we illustrate with an example. Suppose a harmonic wave travels from P_1 to P_2 through several different homogeneous, isotropic media, as shown in Figure 5.30. Then the optical path is

$$\ell_{P_1P_2} = n_1z_1 + n_2z_2 + n_3z_3 \tag{5.107}$$

This equation has an important interpretation when we use the definition of the index of refraction that we gave at the beginning of Chapter 1: applying this definition we write

$$n_1 = \frac{c}{v_1} \qquad n_2 = \frac{c}{v_2} \qquad n_3 = \frac{c}{v_3} \tag{5.108}$$

and, since time equals distance divided by speed,

$$t_1 = \frac{z_1}{v_1} \qquad t_2 = \frac{z_2}{v_2} \qquad t_3 = \frac{z_3}{v_3} \tag{5.109}$$

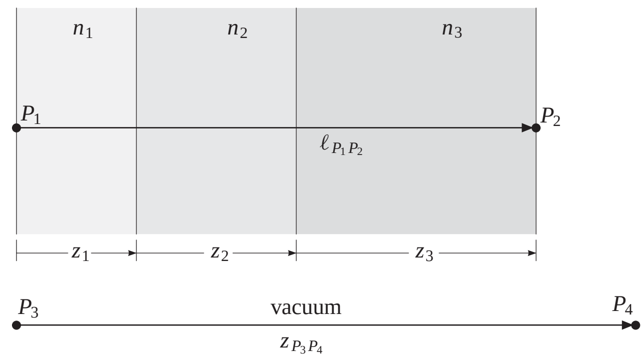

Figure 5.30

To obtain another expression for $\ell_{P_1P_2}$ in Equation 5.107, we first substitute Equations 5.108 into Equation 5.107, and second, successively solve each of equations in Equation 5.109 for z_1, z_2, z_3 before substituting into Equation 5.107; we get

$$\begin{aligned} \ell_{P_1P_2} &= n_1z_1 + n_2z_2 + n_3z_3 \\ &= \left(\frac{c}{v_1}\right)(v_1t_1) + \left(\frac{c}{v_2}\right)(v_2t_2) + \left(\frac{c}{v_3}\right)(v_3t_3) \\ &= ct_1 + ct_2 + ct_3 \\ &= c(t_1 + t_2 + t_3) = ct_{\text{tot}} = z_{P_3P_4} \end{aligned} \tag{5.110}$$

Inspecting this equation, we observe that the time t_{tot} is the time for the light to travel through the media from points P_1 to P_2. When we multiply this total time by the speed of light in a vacuum, we get the distance $z_{P_3P_4}$ (see Figure 5.30). That is, Equation 5.110 says that the optical path in several media (or one medium) is equal to the geometric path traveled by light in a vacuum—the times to travel the separate paths are the same.

In Figure 5.30, the angles of incidence and refraction are zero, but Equation 5.110 holds for nonzero angles as well. Even when there is a continuous change in the index of refraction for a medium, an equation similar to Equation 5.110 can be obtained.

PROBLEMS

Note: Angles with no units are in radians. For the meaning of default angle, see the note under Equation 5.6b.

5.1 The following expressions are complex numbers in rectangular form:

(a) $5+i6$ (b) $-5+i3$

(c) $2-i2\sqrt{3}$ (d) $-\sqrt{6}-i\sqrt{2}$

Write each of these expressions in polar form with the default angle in both radians and degrees, and draw a phasor diagram.

5.2 The following expressions are complex numbers written in polar form (where there is no degree unit, the angle is in radians):

(a) $6\,e^{i\,10^\circ}$ (b) $4\,e^{i\,125^\circ}$

(c) $2\,e^{i\,(-3\pi/8)}$ (d) $14.7\,e^{i\,(-135^\circ)}$

Write each of these expressions in rectangular form and draw a phasor diagram.

5.3 Suppose $z_1 = -5+i5$ and $z_2 = 2+i2\sqrt{3}$, calculate (a) z_1+z_2, (b) z_1-z_2, (c) z_1z_2, (d) z_1/z_2. Express the results in rectangular form.

5.4 Suppose $z_1 = 6\,e^{i\,(-120^\circ)}$ and $z_2 = 4\,e^{i\,(-45^\circ)}$, determine (a) z_1+z_2, (b) z_1-z_2, (c) z_1z_2, (d) z_1/z_2. Express the results in polar form with the angle in default form in both radians and degrees.

5.5 Suppose $z = -5+i6$. Determine (a) the magnitude using Equation 5.29 (that is, calculate $\sqrt{zz^\star}$), (b) the real part using Equation 5.30a, and (c) the imaginary part using Equation 5.30b.

5.6 Suppose the complex number is given in polar form $z = 10\,e^{i\,(-35^\circ)}$. Determine (a) the magnitude using Equation 5.29 (that is, calculate $\sqrt{zz^\star}$), (b) the real part using Equation 5.30a, and (c) the imaginary part using Equation 5.30b.

5.7 Suppose the complex number is

$$z = \frac{2+i4}{3+i5}$$

Determine (a) the magnitude using Equation 5.29 (that is, calculate $\sqrt{zz^\star}$), (b) the Re part using Equation 5.30a, and (c) the Im part using Equation 5.30b.

5.8 Show the derivation of Equation 5.23 in more detail; some steps are missing.

5.9 Derive Equation 5.26b starting with Equation 5.24.

5.10 Prove Equation 5.29.

5.11 Given that $z_1 = 5+i6$ and $z_2 = 8+i3$, follow Example 5.2.8 as a guide and calculate $\mathrm{Re}(z_1z_2)$ (a) directly, (b) using Equation 5.35, and (c) using Equation 5.36. (d) Then find $\mathrm{Re}(z_1)\,\mathrm{Re}(z_2)$. Is Equation 5.37 obeyed?

5.12 Using the derivation of Equation 5.36 as a guide, show that

$$\mathrm{Im}(z_1z_2) = 2\,\mathrm{Im}(z_1)\,\mathrm{Re}(z_2) + \mathrm{Im}(z1^\star\,z2)$$

5.13 Suppose the equation of a traveling harmonic wave is

$$E = \sqrt{3}\,\cos\left(\frac{3}{2}z - \frac{4}{5\pi}t + 0.01\right)$$

With Equation 5.75 and Figure 5.23 as a guide, determine the (a) amplitude, (b) phase, (c) angular wave number, (d) angular frequency, (e) phase constant, (f) wave number, (g) wavelength, (h) frequency, (i) period, and (j) speed. (k) Is the direction of wave travel in the $+z$ direction or the $-z$ direction? Assuming that the length unit is the m, the time unit is the s, and the E unit is the V/m include units with your answers where appropriate.

5.14 Write the traveling harmonic wave in Problem 5.13 as a complex expression in polar form.

5.15 Take the polar-form expression of Problem 5.14 and calculate the intensity I using Equation 5.85. Include appropriate units with your answer.

5.16 Rederive Equation 5.85 putting in all the steps.

5.17 Suppose a traveling harmonic wave is given by the sum of two traveling harmonic waves in polar form:

$$\mathbf{E} = 4e^{i\,(2z-3t-0.1)} + 5e^{i\,(6z-3t-0.2)}$$

Calculate the intensity I at $z = 7$ m using Equation 5.85.

5.18 Given that

$$\mathbf{E} = E_0\left(1+e^{i\delta}\right)e^{i\omega t}$$

determine the intensity I using Equation 5.85.

5.19 Using the definition of the average of a function given by Equation 5.84, determine the average of the function

$$f(t) = \sin \omega t$$

(where ω is a constant) over the interval from $t_1 = 0$ to $t_2 = \pi/\omega$.

5.20 Under the same conditions as Problem 5.19, calculate the average of the function

$$f(t) = \cos \omega t$$

5.21 Two harmonic waves start out in phase from points O_1 and O_2, respectively, as in Figure 5.27. The waves travel in water, for which $n = 1.33$, to a common point P such that the geometric paths are $z_{O_1P} = 10.29$ nm and $z_{O_2P} = 12.42$ nm. Calculate (a) the optical path difference Δ, and (b) the phase difference δ in both radians and degrees for light of wavelength $\lambda = 632.8$ nm.

5.22 Using a diagram similar to the one in Figure 5.27, two harmonic waves start out in phase from points O_1 and O_2 and travel to the point P. However, on the way to the point P, the waves travel through several different media such that

$$n_{O_1P_1} = 1, \qquad n_{P_1P_2} = 1.33, \qquad n_{P_2P} = 1$$

$$n_{O_2P_3} = 1, \qquad n_{P_3P_4} = 1.50, \qquad n_{P_4P} = 1$$

and

$$z_{O_1P_1} = 20.5\,\text{mm},\ z_{P_1P_2} = 25.5\,\text{mm},\ z_{P_2P} = 19.3\,\text{mm}$$

$$z_{O_2P_3} = 24.5\,\text{mm},\ z_{P_3P_4} = 26.0\,\text{mm},\ z_{P_4P} = 23.4\,\text{mm}$$

Calculate (a) the optical path difference Δ, and (b) the phase difference δ in both radians and degrees for light of wavelength $\lambda = 546.1$ nm.

5.23 In a diagram which is like Figure 5.27, imagine that two waves travel in glass of $n = 1.518$. If $\lambda = 632.8$ nm in a vacuum, calculate (a) the corresponding angular wave number k in a vacuum, (b) the wavelength λ_n in the medium, and (c) the corresponding k_n. The two waves are in phase at O_1 and O_2, and travel to the point P such that $z_{O_1P} = 100.1$ nm and $z_{O_2P} = 110.2$ nm. Determine (d) the optical path difference Δ, and (e) the phase difference δ in rad and deg.

5.24 Suppose a harmonic wave of light travels through several media from point P_1 to P_2, as shown in Figure 5.31. For these two points, calculate (a) the geometrical path $z_{P_1P_2}$, and (b) the optical path $\ell_{P_1P_2}$. Using $c = 3 \times 10^8$ m/s for the speed of light in a vacuum, determine for each of the three media, (c) the speeds of wave travel v_1, v_2, v_3, and (d) the times t_1, t_2, t_3. Then find (e) the total time t_{tot} for the wave to travel from P_1 to P_2, and (f) the distance that light travels in a vacuum in this time of t_{tot}; that is, calculate $z_{P_3P_4} = ct_{tot}$. Is $\ell_{P_1P_2}$ equal to $z_{P_3P_4}$?

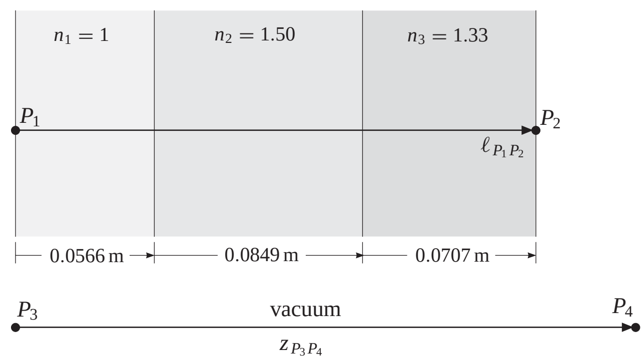

Figure 5.31

Chapter 5: Answers to Problems

5.1 (a) $7.81\, e^{i0.876}$, $7.81\, e^{i50.2^\circ}$; (b) $5.83\, e^{i2.60}$, $5.83\, e^{i149.0^\circ}$; (c) $4\, e^{i(-1.05)}$, $4\, e^{i(-60^\circ)}$; (d) $2.83\, e^{i(-2.62)}$, $2.83\, e^{i(-150^\circ)}$

5.2 (a) $5.91 + i1.04$; (b) $-2.29 + i3.28$; (c) $0.765 - i1.85$; (d) $-10.4 - i10.4$

5.3 (a) $-3 + i8.46$; (b) $-7 + i1.54$; (c) $-27.3 - i7.32$; (d) $0.458 + i1.71$

5.4 (a) $8.03\, e^{i(-1.59)}$, $8.03\, e^{i(-91.2^\circ)}$;
(b) $6.29\, e^{i(-2.76)}$, $6.29\, e^{i(-157.9^\circ)}$;
(c) $24\, e^{i(-2.88)}$, $24\, e^{i(-165^\circ)}$;
(d) $1.5\, e^{i(-1.31)}$, $1.5\, e^{i(-75^\circ)}$

5.5 (a) 7.81; (b) −5; (c) 6

5.6 (a) 10; (b) 8.19; (c) −5.74

5.7 (a) 0.767; (b) 13/17; (c) 1/17

5.11 (a) 22; (b) 22; (c) 22; (d) 40

5.13 (a) $\sqrt{3}\,\mathrm{V/m} = 1.73\,\mathrm{V/m}$; (b) $\frac{3}{2}z - \frac{4}{5\pi}t + 0.01$;
(c) $\frac{3}{2}\,\mathrm{rad/m} = 1.5\,\mathrm{rad/m}$; (d) $\frac{4}{5\pi}\,\mathrm{rad/s} = 0.255\,\mathrm{rad/s}$;
(e) -0.01 rad; (f) $\frac{3}{4\pi}\,\mathrm{m}^{-1} = 0.239\,\mathrm{m}^{-1}$;
(g) $\frac{4\pi}{3}\,\mathrm{m} = 4.19\,\mathrm{m}$; (h) $\frac{2}{5\pi^2}\,\mathrm{Hz} = 0.0405\,\mathrm{Hz}$;
(i) $\frac{5\pi^2}{2}\,\mathrm{s} = 24.7\,\mathrm{s}$; (j) $\frac{8}{15\pi}\,\mathrm{m/s} = 0.170\,\mathrm{m/s}$

5.14 $\sqrt{3}\, e^{i\left(\frac{3}{2}z - \frac{4}{5\pi}t + 0.01\right)}$

5.15 $3C\ \mathrm{W/m^2}$

5.17 $3.77C\ \mathrm{W/m^2}$

5.18 $4CE_0^2 \cos^2(\delta/2)$

5.19 $2/\pi$

5.20 0

5.21 (a) 2.83 nm; (b) 0.0281 rad = 1.61 deg

5.22 (a) 13.2 nm; (b) 0.152 rad = 8.69 deg

5.23 (a) 0.009923 rad/nm; (b) 416.9 nm;
(c) 0.01507 rad/nm; (d) 15.33 nm;
(e) 0.1522 rad = 8.722 deg

5.24 (a) 0.212 m; (b) 0.278 m;
(c) 3.00×10^8 m/s, 2.00×10^8 m/s, 2.26×10^8 m/s;
(d) 0.189 ns, 0.425 ns, 0.313 ns;
(e) 0.927 ns; (f) 0.278 m