Chapter 7—Outline

Fraunhofer Diffraction

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Chapter 7
Fraunhofer Diffraction

7.1 Huygens' Principle

We shall build our theory of diffraction upon Huygens' principle, a model that allows a description of what happens when something that has a wave nature travels through openings or near edges. The basic idea of this principle is to use wavelets that are spherical in nature with each wavelet expanding outward from a point source P'; this point source is often called a secondary source. A wavefront for one of these wavelets is shown in Figure 7.1. The amplitude of any of these wavelets varies inversely with the distance ρ from the secondary source (ρ is also the radius of the spherical wavelet), and varies continuously from a maximum in the forward direction to a minimum of zero in the backward direction, as we have tried to indicate in Figure 7.1 by making the circle heavier where the amplitude is bigger—we shall call this variation with angle around the wavelet, the angle variation. However, in our description of Fraunhofer diffraction—and in the next chapter, Fresnel diffraction—we need only the front part of the wavelets so we ignore the angle variation.

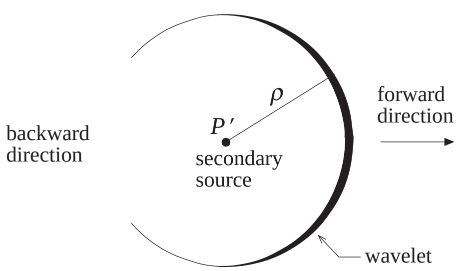

Figure 7.1

The equation of a wavelet expanding outward with time t in the ρ direction is written as a harmonic wave (ignoring the angle variation):

$$E = \frac{A}{\rho} \cos\left(k\rho - \omega t - \phi\right) \tag{7.1}$$

where E is the electric field in units of V/m (or as in Figure 7.2, we use mV/nm), A is the strength of the secondary source P', the amplitude is A/ρ, the angular wave number is k which equals $2\pi/\lambda$, the radius of the wavelet is ρ, the angular frequency is ω which equals $2\pi\nu$ or $2\pi/T$, the time is t, the phase constant is ϕ, and the argument of the cosine function, $(k\rho - \omega t - \phi)$, is the phase—see Figure 5.23 for a summary of the meaning of these parameters.

A graph of Equation 7.1 is shown in Figure 7.2, where we assume $\lambda = 632.8$ nm (the wavelength of He-Ne laser light), $A = 10^4$ mV, $t = 0$ s, and $\phi = 0$ rad. The wavefront of a wavelet, as represented in Figure 7.1, can be imagined to be located on the top of each of the peaks in Figure 7.2. Then as time goes on, these wavelet wavefronts move and expand outward like the outward-moving ripples created by a stone dropped in a quiet pond (of course, the ripples do not have an angle variation).

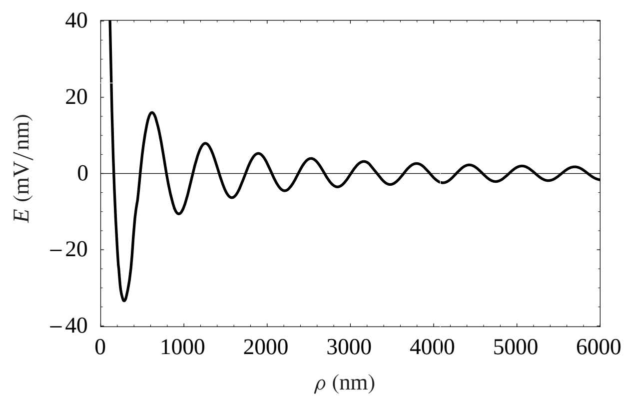

Figure 7.2

As we discussed in Chapter 5, work with a harmonic wave like the one in Equation 7.1 is easier when written as a complex expression in polar form:

$$\mathbf{E} = \frac{A}{\rho} e^{i(k\rho - \omega t - \phi)} \tag{7.2}$$

where the symbol printed in bold type indicates an expression written in the complex plane. If we expand Equation 7.2 into rectangular form, we get

$$\mathbf{E} = \frac{A}{\rho} \cos(k\rho - \omega t - \phi) + i\,\frac{A}{\rho} \sin(k\rho - \omega t - \phi) \tag{7.3}$$

and we see that Equation 7.1 is the real part of the expression in Equation 7.3. It is convenient to manipulate ϕ into the strength A as follows:

$$\mathbf{E} = \frac{A}{\rho} e^{i(k\rho - \omega t)}\, e^{-i\phi} = \frac{A\,e^{-i\phi}}{\rho} e^{i(k\rho - \omega t)} \tag{7.4}$$

We now wish to generalize to many secondary sources using Equation 7.4 as a guide. Suppose there are two secondary sources emitting wavelets, one located at P_1' and the other at P_2', as shown in Figure 7.3. As the wavelets overlap, or superimpose on each other, they interfere. Suppose we want to investigate this interference at the point P, as indicated in Figure 7.3, then we write

$$\mathbf{E}(P) = \frac{A_1\, e^{-i\phi_1}}{\rho_1}\, e^{i(k\rho_1 - \omega t)} + \frac{A_2\, e^{-i\phi_2}}{\rho_2}\, e^{i(k\rho_2 - \omega t)} \qquad (7.5)$$

Clearly, we can generalize Equation 7.5 to any number of secondary sources.

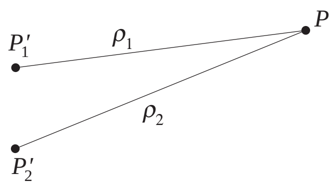

Figure 7.3

We introduce a convenient simplification by assuming that the light leaving the secondary source P_2' is in phase with that leaving P_1' so that we can set $\phi_1 = \phi_2 = 0$; that is, the peaks and troughs of the wavelets leave P_1' and P_2' at the same time. In general, if there are N secondary sources, then $\phi_1 = \phi_2 = \cdots = \phi_N = 0$.

7.2 Light Falling on an Aperture

Let's now imagine that we have light that is monochromatic (one color) and coherent (waves in step with each other) illuminating an opaque screen having one pinhole in it; in general, the incident light can fall on the screen at an angle, as shown in Figure 7.4(a). So that we can use the above mentioned simplification of $\phi_1 = \phi_2 = \cdots = \phi_N = 0$, we work with the simpler case shown in Figure 7.4(b), where the incident wavefronts are parallel to the screen. According to Huygens' principle, we determine what happens on the other side of the screen by placing secondary sources in the pinhole so that wavelets are emitted: the superposition of these wavelets then represents the behavior of monochromatic, coherent light after it passes through the pinhole. However, to build things in smaller steps, we assume that the pinhole is so small that only one secondary source P_1' is needed, as shown in Figure 7.4(b). Then, based on what we said before, we describe the wavelet at the point P as

$$\mathbf{E}_1(P) = \frac{A_1}{\rho_1}\, e^{i(k\rho_1 - \omega t)} \qquad (7.6)$$

Since it is reasonable to assume that the strength A_1 of the secondary source P_1' is proportional to the area $\Delta S_1'$ of the pinhole, we write

$$A_1 = B_1\, \Delta S_1' \qquad (7.7)$$

where B_1 is a real number proportional to the amplitude of the light falling on the screen from the left. Substituting Equation 7.7 into 7.6, we obtain

$$\mathbf{E}_1(P) = \frac{B_1\, \Delta S_1'}{\rho_1}\, e^{i(k\rho_1 - \omega t)} \qquad (7.8)$$

If we make a second pinhole in the screen, as shown in Figure 7.5, then the superposition of the wavelets from both secondary sources at the point P is simply the sum of two expressions like Equation 7.8:

$$\begin{aligned}\mathbf{E}(P) &= \mathbf{E}_1(P) + \mathbf{E}_2(P)\\ &= \frac{B_1\, \Delta S_1'}{\rho_1}\, e^{i(k\rho_1 - \omega t)} + \frac{B_2\, \Delta S_2'}{\rho_2}\, e^{i(k\rho_2 - \omega t)} \qquad (7.9)\end{aligned}$$

If we now make many pinholes in the screen, say N of them, then we get the diagram shown in Figure 7.6, where we have attached explanatory symbols to the 1st, 2nd, and Nth

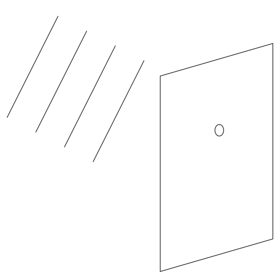

Figure 7.4(a)

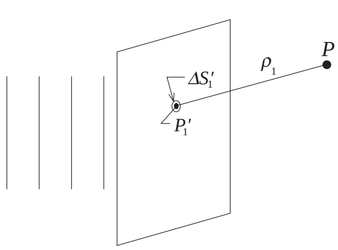

Figure 7.4(b)

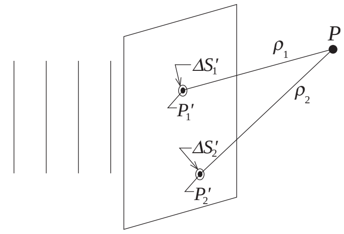

Figure 7.5

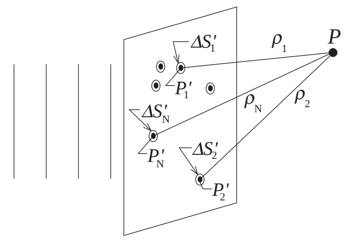

Figure 7.6

pinholes. Based on Equation 7.9, we write the expression that gives the superposition of the wavelets at the point P from all N secondary sources is

$$\begin{aligned}\mathbf{E}(P) &= \mathbf{E}_1(P) + \mathbf{E}_2(P) + \cdots + \mathbf{E}_N(P) \\ &= \frac{B_1\,\Delta S_1'}{\rho_1}\,e^{i(k\rho_1-\omega t)} + \frac{B_2\,\Delta S_2'}{\rho_2}\,e^{i(k\rho_2-\omega t)} \\ &\quad + \cdots + \frac{B_N\,\Delta S_N'}{\rho_N}\,e^{i(k\rho_N-\omega t)} \qquad (7.10)\end{aligned}$$

or more elegantly in terms of the summation symbol

$$\mathbf{E}(P) = \sum_{n=1}^{N} \mathbf{E}(P_n) = \sum_{n=1}^{N} \frac{B_n\,\Delta S_n'}{\rho_n}\,e^{i(k\rho_n-\omega t)} \qquad (7.11)$$

Finally, we rearrange Equation 7.11 into a form that is easier to generalize to the integral form

$$\mathbf{E}(P) = \sum_{n=1}^{N} B_n\,\frac{e^{i(k\rho_n-\omega t)}}{\rho_n}\,\Delta S_n' \qquad (7.12)$$

In a diagram like Figure 7.6, our mental picture of what is happening is that as the incident parallel wavefronts fall on the screen, they make the secondary sources vibrate and emit wavelets; these wavelets expand outward to the right and interfere producing a diffraction pattern. Because a given incident wavefront strikes all secondary sources at the same time, we know that all the secondary sources will vibrate in phase with each other, which is the reason we can set the ϕ's (which we talked about before) to zero. Finally, we imagine that there are an extremely large number of pinholes in the screen, and that they are so close to each other that their openings touch so that we get a continuous opening (or aperture), as shown in Figure 7.7. To use the language of calculus, the variables in Equation 7.12 must be changed slightly: the pinhole area $\Delta S_n'$ becomes an infinitesimal area dS' at the center of which is a secondary source P', and the distance to the point P is called ρ instead of ρ_n (see Figure 7.7). We locate dS' a distance r' from an origin O, which is at any convenient place on the screen or aperture. Recalling that B_n is proportional to the amplitude of the light falling on the screen at the nth pinhole, we can now represent B_n as a function of r', namely $B(r')$. With these changes, Equation 7.12 generalizes to the integral form

$$\mathbf{E}(P) = \int_{S'} B(r')\,\frac{e^{i(k\rho-\omega t)}}{\rho}\,dS' \qquad (7.13)$$

It also helps to understand that the point P in Figure 7.7 is located on an observation screen that is parallel to the plane of the aperture; it is on this screen that the diffraction pattern is viewed. Our goal is to calculate the intensity I at the point P located anywhere on the observation screen.

If we require the light falling on the aperture from the left to be uniformly distributed over the aperture S', then $B(r')$ is no longer a function of r', but is a constant which we simply call B. Factoring this constant out of the integral, Equation 7.13 becomes

$$\mathbf{E}(P) = B\int_{S'} \frac{e^{i(k\rho-\omega t)}}{\rho}\,dS' \qquad (7.14)$$

We are now ready to evaluate the integral in Equation 7.14 for several apertures of different shapes. To make it easier to carry out this evaluation, it is convenient to separate the evaluation into two parts called Fraunhofer diffraction and Fresnel diffraction. In Fraunhofer diffraction, the point P is far from the aperture; whereas, in Fresnel diffraction, it is relatively close.

7.3 Fraunhofer Diffraction by a Single Aperture

7.3.1 The Fraunhofer approximation

To help in the evaluation of Equation 7.14, we redraw Figure 7.7 with coordinate systems in Figure 7.8. As we mentioned before, in Fraunhofer diffraction the point P is on an observation screen far from the aperture S'; in terms of the parameters shown in Figure 7.8, this "far-away" requirement means that quantities such as ρ, r, and D are large compared to the dimensions of the aperture—that is, large compared to any values r' can take as the point P' moves in the aperture.

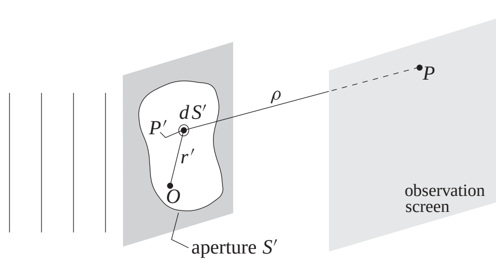

Figure 7.7

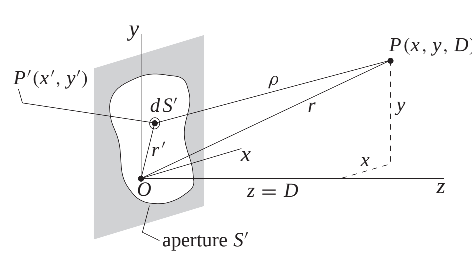

Figure 7.8

To get the expressions we need, it helps to change some of the quantities into vectors, as shown in Figure 7.9. Then

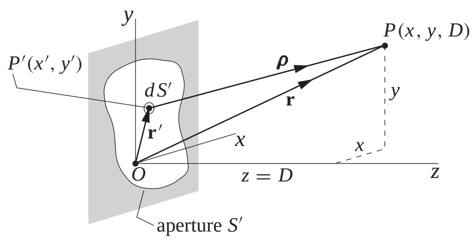

Figure 7.9

by inspection we obtain the vectors $\mathbf{r}$, $\mathbf{r}'$, and $\boldsymbol{\rho}$ in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + D\mathbf{k} \tag{7.15}$$

$$\mathbf{r}' = x'\mathbf{i} + y'\mathbf{j} \tag{7.16}$$

$$\begin{aligned}\boldsymbol{\rho} &= \mathbf{r} - \mathbf{r}' \\ &= \left(x - x'\right)\,\mathbf{i} + \left(y - y'\right)\,\mathbf{j} + D\,\mathbf{k}\end{aligned} \tag{7.17}$$

We next obtain the square of the quantities in Equations 7.15 and 7.16 by calculating the dot product:

$$r^2 = \mathbf{r}\cdot\mathbf{r} = x^2 + y^2 + D^2 \tag{7.18}$$

$$r'^{\,2} = \mathbf{r}'\cdot\mathbf{r}' = x'^2 + y'^2 \tag{7.19}$$

$$\begin{aligned}\rho^2 &= \boldsymbol{\rho}\cdot\boldsymbol{\rho} \\ &= \left(x - x'\right)^2 + \left(y - y'\right)^2 + D^2\end{aligned} \tag{7.20}$$

We expand the squared terms in Equation 7.20, rearrange terms, and substitute Equation 7.18 to obtain

$$\begin{aligned}\rho^2 &= x^2 - 2xx' + x'^2 + y^2 - 2yy' + y'^2 + D^2 \\ &= r^2 - 2\left(xx' + yy'\right) + x'^2 + y'^2\end{aligned} \tag{7.21}$$

The Fraunhofer condition is that x'^2 and y'^2 are small enough compared to the other terms that they can be ignored; that is,

$$x'^2 + y'^2 << \left|r^2 - 2\left(xx' + yy'\right)\right| \tag{7.22}$$

With the use of the Fraunhofer condition in Equation 7.22, Equation 7.21 becomes

$$\begin{aligned}\rho^2 &= r^2 - 2\left(xx' + yy'\right) \\ &= r^2\left[1 - \frac{2\left(xx' + yy'\right)}{r^2}\right] \\ &= r^2\left[1 - \frac{2\left(\frac{x}{r}\,x' + \frac{y}{r}\,y'\right)}{r}\right]\end{aligned} \tag{7.23}$$

We can specify the direction of r in terms of the direction angles A, B, C and the corresponding direction cosines α, β, γ. To help us, we draw the r distance of Figure 7.8 in a box, as shown in Figure 7.10. From this diagram, we read off the following expressions:

$$\alpha = \cos A = \frac{x}{r} \tag{7.24a}$$

$$\beta = \cos B = \frac{y}{r} \tag{7.24b}$$

$$\gamma = \cos C = \frac{D}{r} \tag{7.24c}$$

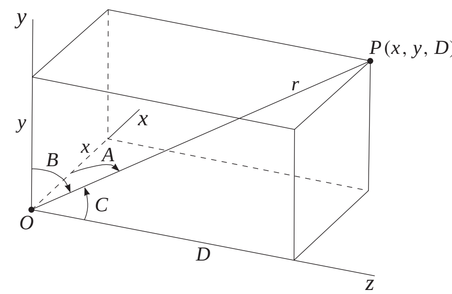

Figure 7.10

We substitute Equations 7.24a and 7.24b into Equation 7.23 to obtain

$$\rho^2 = r^2\left[1 - \frac{2\left(\alpha\,x' + \beta\,y'\right)}{r}\right] \tag{7.25}$$

Remembering that α and β are direction cosines, we have

$$|\alpha| \le 1 \qquad \text{and} \qquad |\beta| \le 1 \tag{7.26}$$

and because $|x'|$ and $|y'|$ are small compared to r, we state

$$\left|2\left(\alpha x' + \beta y'\right)\right| << r \tag{7.27}$$

Therefore, by Equation 7.27, we are allowed to use the binomial theorem to simplify the expression when we take the square root of both sides of Equation 7.25:

$$\begin{aligned}\rho &= r\left[1 - \frac{2\,\left(\alpha\,x' + \beta\,y'\right)}{r}\right]^{1/2} \\ &\approx r\left[1 - \frac{\left(\alpha\,x' + \beta\,y'\right)}{r}\right] \\ &= r - \left(\alpha x' + \beta y'\right)\end{aligned} \tag{7.28}$$

The goal of this section on the Fraunhofer approximation was to obtain a simplified expression for the quantity ρ that appeared in Equation 7.14, which we rewrite and renumber for convenience:

$$\mathbf{E}(P) = B\int_{S'}\frac{e^{i(k\rho - \omega t)}}{\rho}\,dS' \tag{7.29}$$

where ρ is given by Equation 7.28. We see that ρ shows up in two places in Equation 7.29: in the complex exponential and in the denominator. In the complex exponential, ρ is composed of the r term and the much smaller $(\alpha x' + \beta y')$ term: both these terms are important because the complex exponential really represents both a cosine and a sine function (see Equation 5.11)—we shall see how this property develops in the next few pages. But for the ρ in the denominator, the term $(\alpha x' + \beta y')$ is small enough compared to r that it can be ignored. Thus, Equation 7.29 becomes

$$\mathbf{E}(P) = B \int_{S'} \frac{e^{i\{k[r-(\alpha x'+\beta y')]-\omega t\}}}{r}\, dS'$$

$$= B\, \frac{e^{i(kr-\omega t)}}{r} \int_{S'} e^{-ik(\alpha x'+\beta y')}\, dS' \qquad (7.30)$$

where in the last equation we have used the fact that dS' depends only on x' and y' (see Figure 7.8) so the terms involving r and t can be factored out of the integral. Equation 7.30 is as far as we can go until the shape of the aperture S' is given.

7.3.2 A rectangular aperture

As an example of how Equation 7.30 is applied, we choose an aperture S' that is rectangular in shape, as shown in Figure 7.11. Because the aperture is rectangular, it is easiest to evaluate the integral of Equation 7.30 when dS' is also rectangular, as drawn in Figure 7.11; thus

$$dS' = dx'\, dy' \qquad (7.31)$$

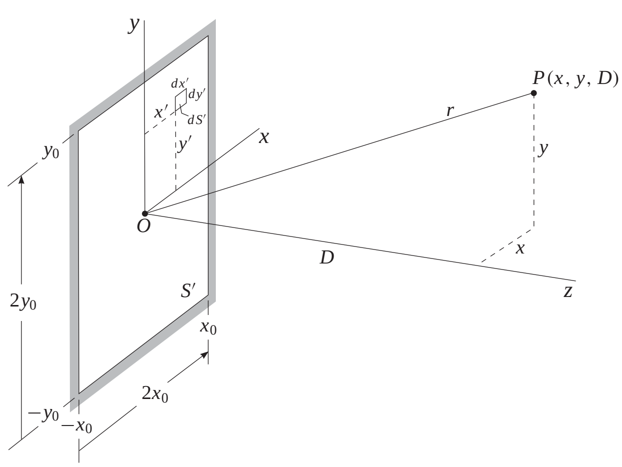

Figure 7.11

Substituting Equation 7.31 into the integral of Equation 7.30, the integral becomes a double integral with the variables of integration x' and y':

$$\mathbf{E}(P) = B\, \frac{e^{i(kr-\omega t)}}{r} \int_{S'} e^{-ik(\alpha x'+\beta y')}\, dS'$$

$$= B\, \frac{e^{i(kr-\omega t)}}{r} \int_{-y_0}^{y_0}\int_{-x_0}^{x_0} e^{-ik(\alpha x'+\beta y')}\, dx'\, dy' \quad (7.32a)$$

We evaluate the integral in Equation 7.32a as follows:

$$\mathbf{E}(P) = B\, \frac{e^{i(kr-\omega t)}}{r} \int_{-y_0}^{y_0}\int_{-x_0}^{x_0} e^{-ik(\alpha x'+\beta y')}\, dx'\, dy'$$

$$= B\, \frac{e^{i(kr-\omega t)}}{r} \int_{-y_0}^{y_0}\int_{-x_0}^{x_0} e^{-ik\alpha x'}\, e^{-ik\beta y'}\, dx'\, dy'$$

$$= B\, \frac{e^{i(kr-\omega t)}}{r} \left(\int_{-x_0}^{x_0} e^{-ik\alpha x'}\, dx'\right)\left(\int_{-y_0}^{y_0} e^{-ik\beta y'}\, dy'\right)$$

$$= B\, \frac{e^{i(kr-\omega t)}}{r} \left[\frac{e^{-ik\alpha x'}}{-ik\alpha}\right]_{-x_0}^{x_0} \left[\frac{e^{-ik\beta y'}}{-ik\beta}\right]_{-y_0}^{y_0}$$

$$= B\, \frac{e^{i(kr-\omega t)}}{r} \left(\frac{-1}{ik\alpha}\right)\left(e^{-ik\alpha x_0} - e^{ik\alpha x_0}\right)$$

$$\left(\frac{-1}{ik\beta}\right)\left(e^{-ik\beta y_0} - e^{ik\beta y_0}\right) \qquad (7.32b)$$

We next expand the exponentials involving x_0 and y_0 in terms of sine and cosine functions by using Euler's identity in Equation 5.11:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad (7.33)$$

Applying this relation to Equation 7.32b, canceling terms where we can, and rearranging, we get

$$\mathbf{E}(P) = e^{i(kr-\omega t)}\, \frac{4B}{r}\, \frac{\sin k\alpha x_0}{k\alpha}\, \frac{\sin k\beta y_0}{k\beta} \qquad (7.34)$$

We put Equation 7.34 into a more symmetric form by multiplying the numerator by $x_0 y_0$, and then divide by the same terms:

$$\mathbf{E}(P) = e^{i(kr-\omega t)}\, \frac{4Bx_0y_0}{r}\, \frac{\sin k\alpha x_0}{k\alpha x_0}\, \frac{\sin k\beta y_0}{k\beta y_0} \qquad (7.35)$$

Finally, what we really want is the intensity I, which was given in Chapter 5 as Equation 5.85; we list it again and give it a new equation number for convenience:

$$I = C\, \mathbf{E}\,\mathbf{E}^\star \qquad (7.36)$$

where C is a real constant. Applying this expression to Equation 7.35, we see that when we take $\mathbf{E}(P)$ and multiply by its complex conjugate, the exponential $e^{i(kr-\omega t)}$ multiplies to one, so that we obtain for the intensity at the point P:

$$I(P) = C\left(\frac{4Bx_0y_0}{r}\right)^2$$

$$\left(\frac{\sin k\alpha x_0}{k\alpha x_0}\right)^2 \left(\frac{\sin k\beta y_0}{k\beta y_0}\right)^2 \qquad (7.37)$$

It is convenient to make one more approximation: looking back at Figure 7.10 (which we redraw as Figure 7.12), we see that if x and y are not made too big, then $r \approx D$. Or more precisely, if we keep D large compared to x and y, then by referring to Figure 7.12 we see that

$$r = \sqrt{x^2 + y^2 + D^2} \approx D \tag{7.38a}$$

$$\alpha = \frac{x}{r} \approx \frac{x}{D} \tag{7.38b}$$

$$\beta = \frac{y}{r} \approx \frac{y}{D} \tag{7.38c}$$

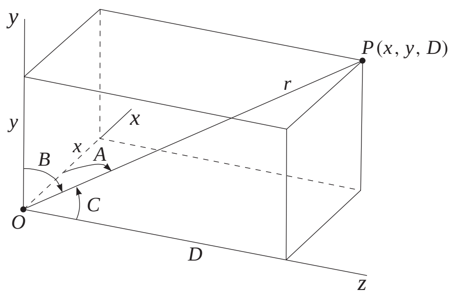

Figure 7.12

Substituting Equations 7.38 into Equation 7.37 gives

$$I(P) = C\left(\frac{4Bx_0y_0}{D}\right)^2 \left(\frac{\sin k\alpha x_0}{k\alpha x_0}\right)^2 \left(\frac{\sin k\beta y_0}{k\beta y_0}\right)^2 \tag{7.39}$$

where it is understood that the α and β that appear in this equation are now given by the right sides of Equations 7.38b and 7.38c. We can simplify this equation by observing that the maximum values of

$$\left(\frac{\sin k\alpha x_0}{k\alpha x_0}\right)^2 \quad \text{and} \quad \left(\frac{\sin k\beta y_0}{k\beta y_0}\right)^2$$

are equal to one, and that these maximum values are obtained as α and β tend toward zero—or looking at Figure 7.12, as the point P moves to the z axis. Calling the maximum value I_m, we rewrite Equation 7.39 as

$$\frac{I(P)}{I_m} = \left(\frac{\sin u}{u}\right)^2 \left(\frac{\sin v}{v}\right)^2 \tag{7.40a}$$

where

$$\left.\begin{aligned} I_m &= \left(\frac{4Bx_0y_0}{D}\right)^2, \quad u = k\alpha x_0\,, \quad v = k\beta y_0\,, \\ k &= \frac{2\pi}{\lambda}\,, \qquad \alpha = \frac{x}{D}\,, \qquad \beta = \frac{y}{D} \end{aligned}\right\} \tag{7.40b}$$

To draw a graph that represents the diffraction pattern that is seen on an observation screen placed parallel to the xy plane a distance D away from the origin O, we first draw Figure 7.13 to help us visualize the setup. The aperture causing the diffraction is rectangular with size $2x_0$ by $2y_0$, and is centered at the origin. As the point $P(x, y)$ moves on the observation screen, we want to graph I/I_m defined in Equations 7.40 as a function of x and y. To obtain the graph, we choose $\lambda = 632.8\,\text{nm}$, $D = 0.8\,\text{m}$, $2x_0 = 0.2\,\text{mm}$, and $2y_0 = 0.4\,\text{mm}$; we then use *Mathematica* to draw the three-dimensional graph of I/I_m along the vertical versus x and y on the horizontal plane. We see that I/I_m has a maximum of one when the point P is on the z axis where both x and y equal zero, which is opposite the center of the rectangular aperture. As the point P moves away from the z axis, the value of I/I_m drops off to zero giving a shape for this central diffraction peak (also called the principal maximum) that is rectangular at the base, and, for typical values of λ and D, is normally larger than the size of the aperture; in fact, the smaller we make the aperture, the larger the base of the central diffraction peak becomes. In addition, there are much smaller secondary maxima that appear along the x and y axes, and then smaller yet secondary maxima that are located in the four quadrants—so small that they do not show on this graph.

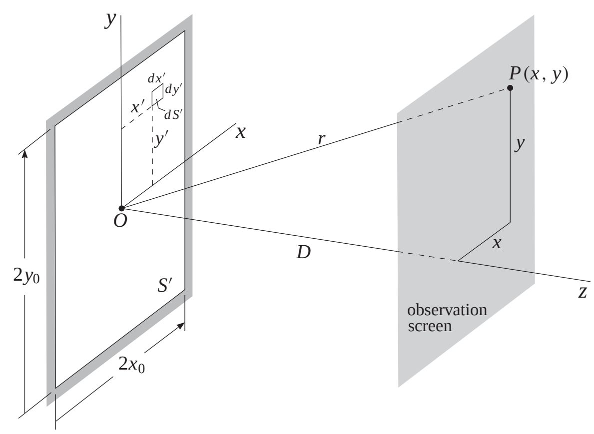

Figure 7.13

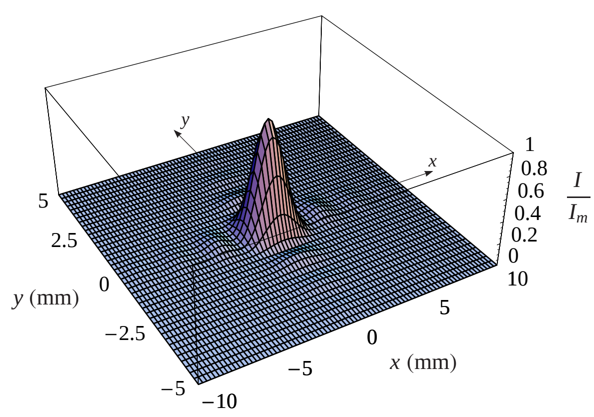

Figure 7.14

To see the secondary maxima more clearly, we redraw the graph without a mesh and allow I/I_m to run from 0 to only 0.05 in Figure 7.15. The principal maximum is now cut off, but the on-axis secondary maximum clearly show, and we discern faint hints of the off-axis secondary maxima in the quadrants. It is important to note the large variation between the values of the principal and the secondary maxima.

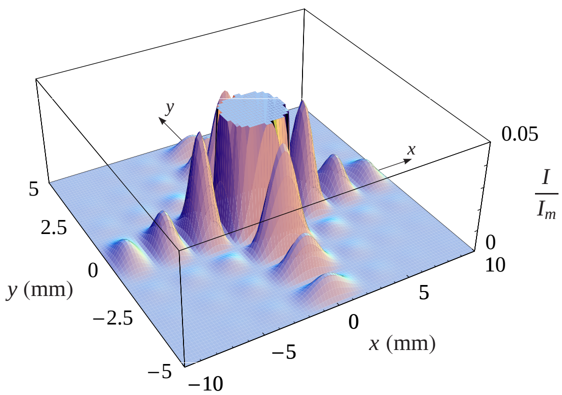

Figure 7.15

To learn more about this diffraction pattern, we investigate where the maxima and minima occur along the x axis, and along lines parallel to this axis. We work with just the x axis because symmetry shows that the formulas for the x axis are also true for the y axis. To perform this procedure, we set $y = \text{constant}$, which makes $v = \text{constant}$, which in turn makes $(\sin v/v)^2 = \text{constant} = c_v$; thus, Equation 7.40a becomes

$$\left(\frac{I}{I_m}\right)_{v=\text{constant}} = c_v \left(\frac{\sin u}{u}\right)^2 \tag{7.41}$$

where we drop the P notation in $I(P)$ for simplicity, and where the meaning of u is given by Equation 7.40b. Calculus provides a convenient way to find the maxima and minima—just differentiate Equation 7.41 with respect to u and set the resulting expression to zero; the solutions to this equation give the values of u where the maxima and minima occur:

$$\frac{d}{du}\left(\frac{I}{I_m}\right)_{v=\text{constant}} = c_v \frac{d}{du}\left(\frac{\sin u}{u}\right)^2 = 0 \tag{7.42}$$

or

$$\frac{d}{du}\left(\frac{\sin u}{u}\right)^2 = 0$$

$$2\left(\frac{\sin u}{u}\right)\left(\frac{\cos u}{u} - \frac{\sin u}{u^2}\right) = 0 \tag{7.43a}$$

We divide both sides of the Equation 7.43a by $\cos u$, use the trig identity $\sin u/\cos u = \tan u$ to get

$$2\left(\frac{\sin u}{u}\right)\left(\frac{1}{u} - \frac{\tan u}{u^2}\right) = 0$$

and then find the common denominator to obtain

$$2\left(\frac{\sin u}{u}\right)\left(\frac{u - \tan u}{u^2}\right) = 0 \tag{7.44b}$$

Equation 7.44b is satisfied in two nontrivial ways:

$$\frac{\sin u}{u} = 0 \tag{7.45}$$

and

$$u - \tan u = 0 \tag{7.46}$$

We determine the solutions of Equation 7.45 first. We see that

$$\frac{\sin u}{u} = 0$$

$$\sin u = 0 \quad \text{for} \quad u \neq 0$$

$$u = \pm\pi, \pm 2\pi, \ldots$$

$$u = \pm m\pi, \quad m = 1, 2, \ldots \tag{7.47}$$

where m is called an order number. If we substitute the u values of Equation 7.47 into Equation 7.41, we see that they make $(I/I_m)_{v=\text{constant}}$ equal to zero; therefore, Equation 7.47 gives the u values for which minima occur. In terms of x, we use the appropriate expressions in Equation 7.40b to get

$$u = \pm m\pi$$

$$k\alpha x_0 = \pm m\pi$$

$$\frac{2\pi}{\lambda}\frac{x}{D} x_0 = \pm m\pi$$

$$x = \pm m \frac{\lambda D}{2x_0}, \quad m = 1, 2, \ldots \tag{7.48}$$

Equation 7.48 gives the values of x where I/I_m in Figures 7.14 or 7.15 has its zeros on the x axis or along lines parallel to the x axis. By symmetry, we find the zeros on the y axis, or along lines parallel to the y axis, to be given by a similar equation (the n in the equation below is also called an order number, just like the m above):

$$y = \pm n \frac{\lambda D}{2y_0}, \quad n = 1, 2, \ldots \tag{7.49}$$

Clearly, from Equations 7.48 and 7.49, the position of the zeros is inversely proportional to $2x_0$ and $2y_0$; that is, inversely proportional to the width and height of the aperture through which the light is passing. The lines

$$x = \pm\lambda D/2x_0 \quad \text{and} \quad y = \pm\lambda D/2y_0 \tag{7.50}$$

give the first set of minima (zeros), and intersect at four points making a rectangle that defines the shape of the principal or central maximum at the base (see Figures 7.14 or 7.15); the same shape as the aperture. Clearly, the expressions in Equation 7.50 show that the size of this rectangle is inversely proportional to $2x_0$ and $2y_0$; therefore, as we mentioned before, the size of the central maximum is inversely proportional to the size of the aperture.

Using the values that we chose earlier of $\lambda = 632.8\,\text{nm}$, $D = 0.8\,\text{m}$, $2x_0 = 0.2\,\text{mm}$, $2y_0 = 0.4\,\text{mm}$, we calculate the positions of eight minima in the x direction of Figures 7.14 or 7.15 as

$$\left.\begin{aligned} x_1 &= \pm\ 2.5312\,\text{mm} \\ x_2 &= \pm\ 5.0624 \\ x_3 &= \pm\ 7.5936 \\ x_4 &= \pm 10.1248 \end{aligned}\right\} \tag{7.51a}$$

The corresponding minima along the y direction are

$$\left.\begin{aligned} y_1 &= \pm 1.2656\,\text{mm} \\ y_2 &= \pm 2.5312 \\ y_3 &= \pm 3.7968 \\ y_4 &= \pm 5.0624 \end{aligned}\right\} \tag{7.51b}$$

We have now found the solutions to Equation 7.45, and investigated their meaning. Next, we search for the solutions to Equation 7.46, namely $u - \tan u = 0$. This equation is more difficult to solve, because it is a transcendental equation; that is, we cannot solve for u using standard algebraic techniques. However, it is possible to solve for u using either a graphical or a numerical method: we use the graphical method first.

To find the values of u that satisfy Equation 7.46 by the graphical method, we first rewrite the equation as

$$\tan u = u \tag{7.52}$$

Next, we define two new functions:

$$w_1 = \tan u \tag{7.53a}$$
$$w_2 = u \tag{7.53b}$$

and then graph these two functions on the same set of axes as shown in Figure 7.16. Because of the repetitive nature of the $w_1 = \tan u$ graph, the $w_2 = u$ straight line intersects the $w_1 = \tan u$ curves many times (really, an infinite number of times), as shown by the solid dots; each intersection dot represents a solution of $\tan u = u$. We note that one solution is $u = 0$, which corresponds to the central maximum (but see Problem 7.5). With the exceptions of $\pm\pi/2$, the other solutions approximate odd multiples of $\pm\pi/2$, and get closer to these odd multiples as $|u|$ gets larger. When these values of u are substituted into the I/I_m expression of Equation 7.41, they yield the maxima. In summary, the solutions of Equation 7.46 are

$$\left.\begin{aligned} &u = 0 \\ &u \approx \pm m\frac{\pi}{2}, \quad m = 3, 5, \ldots \end{aligned}\right\} \tag{7.54}$$

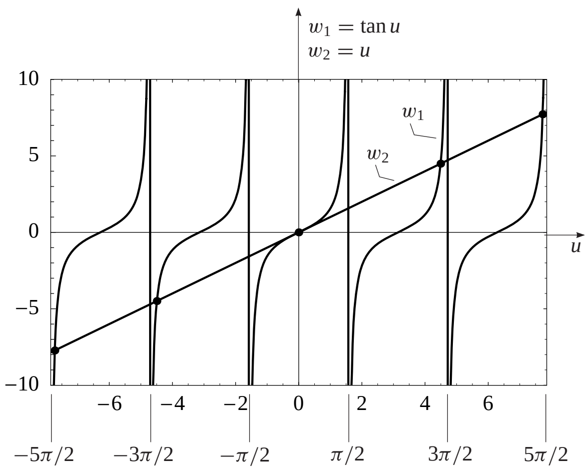

Figure 7.16

We see that the graphical method has limited accuracy. However, an approach that gives better accuracy is a numerical procedure called the bisection method: it is illustrated by the graph in Figure 7.17. The flow chart in Figure 7.18 shows how it is implemented. Referring to the graph of Figure 7.17, the goal is to find the intersection point marked by u; that is, to find the value u. The first step is to estimate or to use the graph of Figure 7.16 to determine u_{lo} and u_{hi} values that lie on either side of the intersection point, but it is important not to set u_{hi} equal to odd multiples of $\pm\pi/2$ because $\tan u$ "blows up" at these points. Once u_{lo} and u_{hi} have been chosen, then the next step in the flow chart is to calculate u_{mid} as the bisection of the interval between u_{lo} and u_{hi}; the value of u_{mid} eventually approaches the desired value of u. Then, continuing to look at the flow chart, the values w_1, w_2, and w_{diff} are calculated. Looking at the graph in Figure 7.17, we see that $w_{\text{diff}} = w_2 - w_1$ approaches zero as we get closer to the desired value of u, and we check for "close enough" by comparing $|w_{\text{diff}}|$ with some small number that we choose. If $|w_{\text{diff}}|$ is still too big, then the key to the method is to note that to the left of the intersection $w_{\text{diff}} = w_2 - w_1$ is positive, while to the right it is negative. As the flow chart indicates, if $w_{\text{diff}} > 0$, then we choose a new value for u_{lo} that equals u_{mid} (if $w_{\text{diff}} < 0$, then u_{hi} would be set equal to u_{mid}). Following the arrows in the flow chart, a new u_{mid} value is calculated, and the steps in the flow chart are repeated. Thus, by choosing new u_{lo} and u_{hi} values as indicated in the flow chart, we squeeze down on the intersection point: we obtain a numerical value of u with an accuracy that depends on how many times we loop through the flow chart and on how many digits we use to represent the numbers.

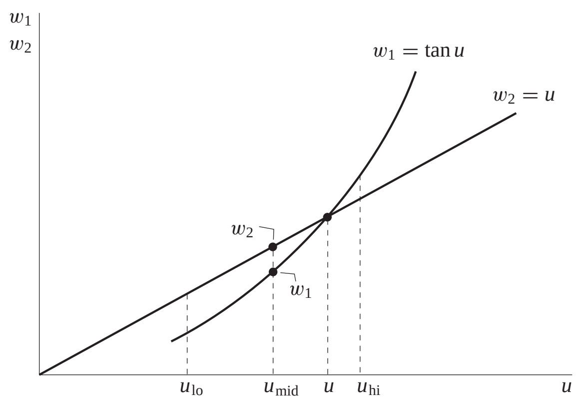

Figure 7.17 Graph illustrating the bisection method.

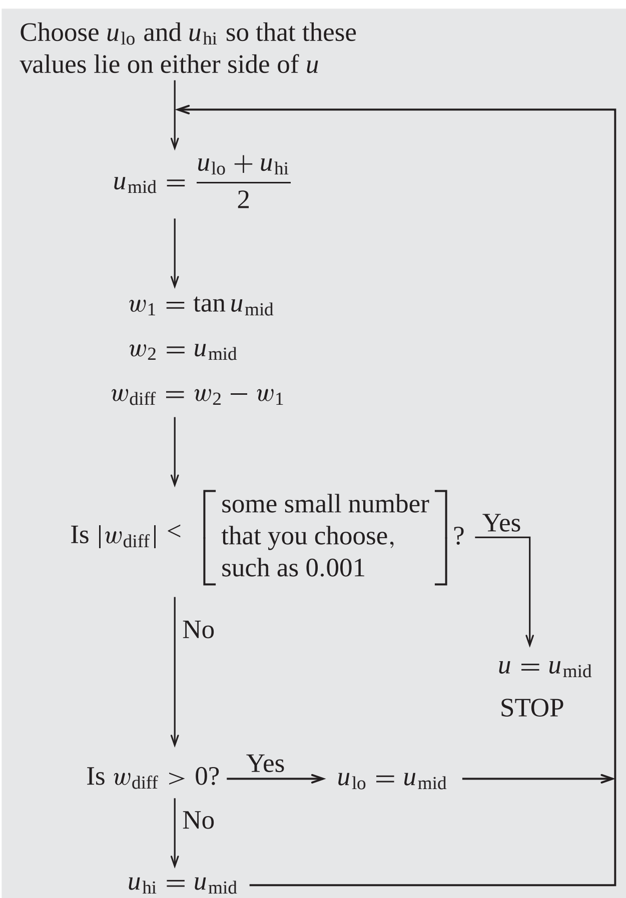

Figure 7.18 Flow chart for the bisection method.

The bisection method outlined by the flow chart in Figure 7.18 can be implemented on a calculator; if the calculator is programmable, then it is yet easier to execute. A program written for computer use can also implement the flow chart. Since the I/I_m graph in Figure 7.15 shows seven maxima, we calculate that many values of u:

$$\left.\begin{aligned} u_0 &= 0 \\ u_1 &= \pm 4.493 = \pm 2.861\,\frac{\pi}{2} \\ u_2 &= \pm 7.725 = \pm 4.918\,\frac{\pi}{2} \\ u_3 &= \pm 10.904 = \pm 6.942\,\frac{\pi}{2} \end{aligned}\right\} \tag{7.55}$$

where we have also determined the results as multiples of $\pi/2$ so that we can see how close we are to the odd multiples of $\pi/2$. We note that these multiples are indeed close to the odd numbers, and get closer as the values of $|u|$ get larger. Referring back to graph of Figure 7.16, we see that these values are in agreement with the ones indicated by the solid dots (only five solutions are shown in the graph to make it less crowded).

We can get the corresponding values of x for the seven maxima along the x axis in Figure 7.15 by looking back at Equation 7.40b and picking out

$$u = k\alpha x_0 = \frac{2\pi}{\lambda}\,\frac{x}{D}\,x_0$$

and then solving for x in terms of u:

$$x = \frac{\lambda D}{\pi\, 2x_0}\,u \tag{7.56}$$

Using the same values that we chose earlier of $\lambda = 632.8$ nm, $D = 0.8$ m, $2x_0 = 0.2$ mm, and $2y_0 = 0.4$ mm, we obtain

$$\left.\begin{aligned} x_0 &= \quad 0 \quad \text{mm} \\ x_1 &= \pm 3.620 \\ x_2 &= \pm 6.224 \\ x_3 &= \pm 8.786 \end{aligned}\right\} \tag{7.57}$$

The subscripts are the order numbers for the maxima; thus, the zeroth order maximum corresponds to x_0, the first order maxima correspond to x_1 (the ± 3.620 mm is said to give the positions of the first order pair of maxima), etc.

The symmetry in the equations for u and v, and x and y, means that the same procedure is used to find the maxima for v and y. Thus, the maxima for v are the same as for u in Figure 7.16 and Equation 7.55. The analog to Equation 7.56 is

$$y = \frac{\lambda D}{\pi\, 2y_0}\,v \tag{7.58}$$

Because $2y_0$ is not equal to $2x_0$, we get values for y that are different than those for x in Equation 7.57. Thus, implementing Equation 7.58, we get

$$\left.\begin{aligned} y_0 &= \quad 0 \quad \text{mm} \\ y_1 &= \pm 1.810 \\ y_2 &= \pm 3.112 \\ y_3 &= \pm 4.393 \end{aligned}\right\} \tag{7.59}$$

What we said about the order numbers for x is also true for y; that is, the subscripts are the order numbers for the maxima: the zeroth order maximum corresponds to y_0, the first order maxima correspond to y_1 (the ± 1.810 mm is said to give the positions of the first order pair of maxima), etc.

The secondary maxima that barely show in the quadrants of Figure 7.15 are found by pairing up the above nonzero x and y coordinates:

$$\left.\begin{matrix} (x_1, y_1), & (x_1, y_2), & (x_1, y_3), & \dots \\ (x_2, y_1), & (x_2, y_2), & (x_2, y_3), & \dots \\ \vdots & \vdots & \vdots & \end{matrix}\right\} \tag{7.60}$$

7.3.3 A circular aperture

We now want to obtain the diffraction pattern for an aperture S' that is circular in shape, as shown in Figure 7.19.

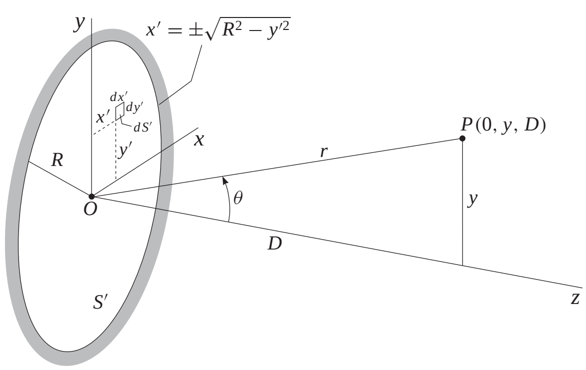

Figure 7.19

The integral we have to evaluate is Equation 7.30, which we list again for convenience:

$$\mathbf{E}(P) = B\,\frac{e^{i(kr-\omega t)}}{r}\int_{S'} e^{-ik(\alpha x'+\beta y')}\,dS' \tag{7.61a}$$

where

$$k = \frac{2\pi}{\lambda}, \qquad \alpha = \frac{x}{r}, \qquad \beta = \frac{y}{r}. \tag{7.61b}$$

Because of the symmetry of the circular aperture, we can choose the point P in the yz plane with the coordinates $(0, y, D)$ to make the evaluation of the integrals simpler; this choice means

$$x = 0 \qquad \text{and} \qquad \alpha = \frac{x}{r} = 0 \tag{7.62}$$

Even though the symmetry of the circle would suggest that we should not use a rectangular dS', we will do so to obtain a particular form for our final result; thus, we write

$$dS' = dx'\,dy' \tag{7.63}$$

Substituting the dS' of Equation 7.63, and the $\alpha = 0$ value of Equation 7.62, into the integral of Equation 7.61a, we obtain a double integral:

$$\int_{S'} e^{-ik(\alpha x'+\beta y')}\,dS' = \int_{-R}^{R}\int_{-\sqrt{R^2-y'^2}}^{\sqrt{R^2-y'^2}} e^{-ik\beta y'}\,dx'\,dy' \tag{7.64a}$$

Evaluating the inner integral of Equation 7.64a first, we get

$$\int_{-\sqrt{R^2-y'^2}}^{\sqrt{R^2-y'^2}} e^{-ik\beta y'}\,dx' = e^{-ik\beta y'}\int_{-\sqrt{R^2-y'^2}}^{\sqrt{R^2-y'^2}} dx' = 2\,e^{-ik\beta y'}\sqrt{R^2-y'^2} \tag{7.64b}$$

The result in Equation 7.64b now becomes the integrand of the outer integral in Equation 7.64a:

$$\int_{-R}^{R} 2e^{-ik\beta y'}\sqrt{R^2-y'^2}\,dy' = 2R\int_{-R}^{R} e^{-i(kR\beta)(y'/R)}\sqrt{1-y'^2/R^2}\;dy' = 2R\int_{-R}^{R} e^{-iv(y'/R)}\sqrt{1-y'^2/R^2}\;dy' \tag{7.64c}$$

where, using the diagram in Figure 7.19 and Equation 7.61b,

$$v = kR\beta = k\,R\,\frac{y}{r} = kR\sin\theta \tag{7.64d}$$

If we change the variable of integration by setting $p = y'/R$; then $dp = dy'/R$, and the limits of the integral change from $-R$ and R to -1 and 1; the integral of Equation 7.64c is now written as

$$2R\int_{-R}^{R} e^{-iv(y'/R)}\sqrt{1-y'^2/R^2}\;dy' = 2R^2\int_{-1}^{1} e^{-ivp}\sqrt{1-p^2}\;dp = 2R^2\int_{-1}^{1}(\cos vp - i\,\sin vp)\sqrt{1-p^2}\;dp \tag{7.65}$$

where in the last step we have expanded the complex exponential with Euler's identity of Equation 5.11.

We now want to expand the integral in Equation 7.65 into two integrals, and then by inspection pick out which portions of the integrands are even or odd functions; this procedure will allow us to set one of the integrals to zero, and to simplify the limits of the other:

$$2R^2 \int_{-1}^{1} (\cos vp - i\ \sin vp) \sqrt{1-p^2}\ dp$$

$$= 2R^2 \int_{-1}^{1} \underbrace{\sqrt{1-p^2}}_{\text{even}}\ \underbrace{\cos vp}_{\text{even}}\ dp$$

$$-i\, 2R^2 \int_{-1}^{1} \underbrace{\sqrt{1-p^2}}_{\text{even}}\ \underbrace{\sin vp}_{\text{odd}}\ dp \qquad (7.66)$$

The first integral to the right of the equals sign in Equation 7.66 has an integrand that is the product of two even functions; therefore, the integrand is an even function which means that the lower limit can be changed to zero and the value of integral doubled. However, the second integral has an integrand that is composed of the product of an even and an odd function; therefore, the integrand is an odd function which means that the value of the integral is zero. Thus, Equation 7.66 becomes

$$2R^2 \int_{-1}^{1} (\cos vp - i\ \sin vp) \sqrt{1-p^2}\ dp$$

$$= 4R^2 \int_{0}^{1} \sqrt{1-p^2}\ \cos vp\, dp \qquad (7.67)$$

Finally, we want to change the coefficient in front of the integral so that we can obtain an expression which we can identify as a special function called the first-order Bessel function:

$$4R^2 \int_{0}^{1} \sqrt{1-p^2}\ \cos vp\, dp$$

$$= 4R^2\, \frac{\pi}{2v}\, \frac{2v}{\pi} \int_{0}^{1} \sqrt{1-p^2}\ \cos vp\, dp$$

$$= \frac{2\pi R^2}{v}\, \frac{2v}{\pi} \int_{0}^{1} \sqrt{1-p^2}\ \cos vp\, dp$$

$$= \frac{2\pi R^2}{v}\, J_1(v) \qquad (7.68)$$

where

$$J_1(v) = \frac{2v}{\pi} \int_{0}^{1} \sqrt{1-p^2}\ \cos vp\, dp \qquad (7.69)$$

is called the first-order Bessel function.

Looking back at what we have accomplished, we see that we have evaluated the integral on the left in Equation 7.64a to be the result in Equation 7.68, namely

$$\int_{S'} e^{-ik(\alpha x' + \beta y')}\, dS' = \frac{2\pi R^2}{v}\, J_1(v) \qquad (7.70)$$

Substituting Equation 7.70 into Equation 7.61a, we have finally found $\mathbf{E}$ at the point P in Figure 7.19:

$$\mathbf{E}(P) = B\, \frac{e^{i(kr-\omega t)}}{r} \int_{S'} e^{-ik(\alpha x' + \beta y')}\, dS'$$

$$= B\, \frac{e^{i(kr-\omega t)}}{r}\, \frac{2\pi R^2}{v}\, J_1(v) \qquad (7.71)$$

Finally, we want the intensity I at the point P, which we calculate in the usual way (see Equation 7.36):

$$I(P) = C\, \mathbf{E}\, \mathbf{E}^{\star}$$

$$= C \left[B\, \frac{e^{i(kr-\omega t)}}{r}\, \frac{2\pi R^2}{v}\, J_1(v) \right] \left[B\, \frac{e^{-i(kr-\omega t)}}{r}\, \frac{2\pi R^2}{v}\, J_1(v) \right]$$

$$= C \left(\frac{B\, 2\pi R^2}{r} \right)^2 \left[\frac{J_1(v)}{v} \right]^2$$

$$\approx C \left(\frac{B\, 2\pi R^2}{D} \right)^2 \left[\frac{J_1(v)}{v} \right]^2 \qquad (7.72)$$

where we have replaced r with D in the last step since these two quantities are approximately equal.

The symmetry of the circular aperture indicates that the principal maximum for $I(P)$ occurs when P is on the z axis where $y = 0$ (see Figure 7.19), or by Equation 7.64d, $v = 0$; we call this maximum value I_m. To obtain this value requires some work. We start the evaluation with Equation 7.72 and then substitute Equation 7.69:

$$I_m = \lim_{v \to 0} I(P) = \lim_{v \to 0} C \left(\frac{B\, 2\pi R^2}{D} \right)^2 \left[\frac{J_1(v)}{v} \right]^2$$

$$= \lim_{v \to 0} C \left(\frac{B\, 2\pi R^2}{D} \right)^2 \left[\frac{1}{v}\, \frac{2v}{\pi} \int_{0}^{1} \sqrt{1-p^2}\ \cos vp\, dp \right]^2 \qquad (7.73a)$$

Canceling the vs in front of the integral, and then allowing $v \to 0$ in the $\cos vp$ term, we obtain

$$I_m = C \left(\frac{B\, 2\pi R^2}{D} \right)^2 \left(\frac{4}{\pi^2} \right) \left[\int_{0}^{1} \sqrt{1-p^2}\, dp \right]^2 \qquad (7.73b)$$

The integral inside the brackets we can evaluate by using *Mathematica* or by looking it up in a table of integrals, and then substituting the limits of 1 and 0; we find

$$\int_0^1 \sqrt{1-p^2}\,dp = \left[\frac{p}{2}\sqrt{1-p^2} + \frac{1}{2}\sin^{-1} p\right]_0^1$$
$$= \frac{\pi}{4} \tag{7.73c}$$

Substituting Equation 7.73c into Equation 7.73b, we get

$$I_m = C\left(\frac{B\,2\pi R^2}{D}\right)^2\left(\frac{1}{4}\right) \tag{7.73d}$$

Before we summarize our equations, we would like to generalize the diagram of Figure 7.19, which we used in developing the preceding equations. In that diagram, the point P is in the yz plane, but symmetry about the z axis says that the same results hold as the point P is rotated about the z axis a distance y away. We name this distance more generally as the radius s, and draw the diagram in Figure 7.20. We then see that $s = \sqrt{x^2+y^2}$, and the β in Equation 7.61b becomes s/r, or s/D, since $r \approx D$. Now we summarize using

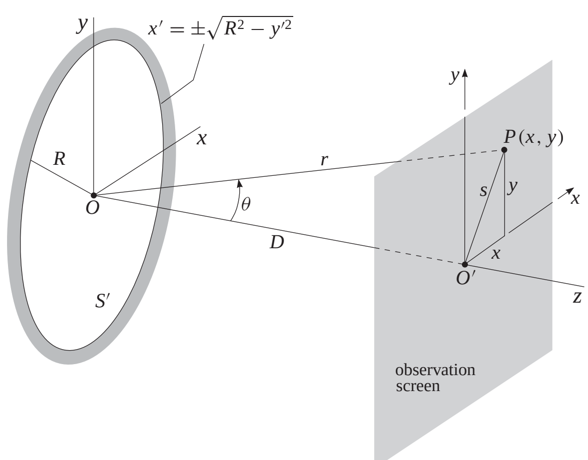

Figure 7.20

a format similar to the one used with the rectangular aperture (see Equations 7.40); thus, using Equation 7.73d in Equation 7.72, we get

$$\frac{I(P)}{I_m} = 4\left[\frac{J_1(v)}{v}\right]^2 \tag{7.74a}$$

where

$$\left.\begin{aligned} J_1(v) &= \frac{2v}{\pi}\int_0^1 \sqrt{1-p^2}\cos vp\,dp\,, \\ I_m &= \frac{C}{4}\left(\frac{B\,2\pi R^2}{D}\right)^2, \quad k = \frac{2\pi}{\lambda}\,, \\ v &= kR\beta, \quad \beta = \frac{s}{D}\,, \quad s = \sqrt{x^2+y^2} \end{aligned}\right\} \tag{7.74b}$$

To get a feeling for the first-order Bessel function $J_1(v)$, we use *Mathematica* to draw its graph between $v = 0$ and $v = 10$, and then display it in Figure 7.21; we also list some values of $J_1(v)$ in Figure 7.22. We observe that this function is a little like a sine function, except the amplitude decreases as v gets larger. The first-order Bessel function $J_1(v)$ is just one of a very large family of Bessel functions; they arise in many applied problems in physics and engineering.

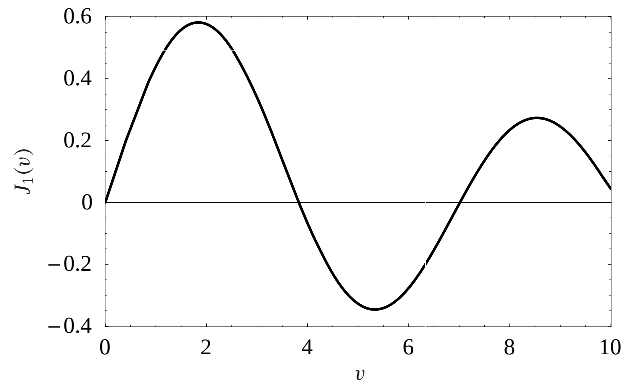

Figure 7.21

v	$J_1(v)$
0	0
0.5	0.242268
1.	0.440051
1.5	0.557937
2.	0.576725
2.5	0.497094
3.	0.339059
3.5	0.137378
4.	−0.066043
4.5	−0.231060
5.	−0.327579
5.5	−0.341438
6.	−0.276684
6.5	−0.153841
7.	−0.004683
7.5	0.135248
8.	0.234636
8.5	0.273122
9.	0.245312
9.5	0.161264
10.	0.043473

Figure 7.22

It would be nice if a pocket calculator had a $J_1(v)$ key like the sin and cos keys, but most calculators do not have this key; thus, it is more difficult to determine values for $J_1(v)$. We have to resort to graphs or tables in handbooks that give the information like that shown in Figures 7.21 and 7.22. Or we can use a computer program like *Mathematica*, which is what we shall do here. To learn more about $J_1(v)$, we determine several of the roots of $J_1(v)$. First of all we see that 0 is a root, and Equations 7.73 show that when $v = 0$, $I(P)$ equals the maximum I_m. By inspecting the graph in Figure 7.21 and the table in Figure 7.22, we pick out three positive roots: the first is between 3.5 and 4.0, the second is between 7.0 and 7.5, and the third is a little larger than 10.0. This information is enough for *Mathematica* to determine these roots; we get

$$\left.\begin{aligned} v_1 &= 3.8317 \\ v_2 &= 7.0156 \\ v_3 &= 10.1735 \end{aligned}\right\} \tag{7.75}$$

These roots are values of v that make $J_1(v)$ equal to zero. Therefore, since Equation 7.74a says that

$$\frac{I}{I_m} = 4\left[\frac{J_1(v)}{v}\right]^2$$

we see that the values in Equation 7.75 give the minima (the zeros) of I/I_m. To see where these minima are located in terms of the radius s of Figure 7.20, we pick out the relations we need from Equation 7.74b:

$$v = kR\beta = \frac{2\pi Rs}{\lambda D} \qquad \text{or} \qquad s = v\,\frac{\lambda D}{\pi d} \tag{7.76}$$

where in the last equation we have replaced $2R$ by the diameter d of the circular aperture (we use the symbol d for diameter, since we have already used D). Substituting the roots of Equation 7.75 into the s relation of Equation 7.76, we obtain (after dividing by π):

$$\left.\begin{aligned} s_1 &= 1.2197\,\frac{\lambda D}{d} \\ s_2 &= 2.2331\,\frac{\lambda D}{d} \\ s_3 &= 3.2383\,\frac{\lambda D}{d} \end{aligned}\right\} \tag{7.77}$$

Or we can express the results of Equation 7.77 in terms of θ. Inspecting Figure 7.20, and since θ is small, we have $\theta \approx s/D$; thus

$$\left.\begin{aligned} \theta_1 &= 1.2197\,\frac{\lambda}{d} \\ \theta_2 &= 2.2331\,\frac{\lambda}{d} \\ \theta_3 &= 3.2383\,\frac{\lambda}{d} \end{aligned}\right\} \tag{7.78}$$

The θ_1 expression in Equation 7.78 with the numeric value rounded to three significant figures, namely $\theta_1 = 1.22\,\lambda/d$, is the relation usually found in beginning books of physics—but the relation is rarely derived, and now we see the reason: a Bessel function is needed.

To obtain numerical values for the radii s where the zeros of I/I_m occur, we choose $\lambda = 632.8\,\text{nm}$, $D = 0.8\,\text{m}$, and $d = 0.2\,\text{mm}$. The expressions in Equation 7.77 become

$$\left.\begin{aligned} s_1 &= 3.087\,\text{mm} \\ s_2 &= 5.653 \\ s_3 &= 8.197 \end{aligned}\right\} \tag{7.79}$$

Using the same data as in the previous paragraph, we draw two 3D graphs with *Mathematica* of I/I_m as a function of x and y, as given by Equations 7.74. In the first graph, Figure 7.23, we let I/I_m run to its maximum value of one. The radius of the circle that corresponds to $I/I_m = 0$ at the base of the central maximum is given by s_1 in Equation 7.79; from Equation 7.77, we see that s_1 is inversely proportional to the diameter d (or radius R) of the circular aperture—that is, as the circular aperture grows smaller the central maximum spreads out; in fact, the entire pattern spreads out. To see the secondary maxima more clearly, we draw the graph in Figure 7.24 with the values of I/I_m restricted to the range from 0 to 0.05.

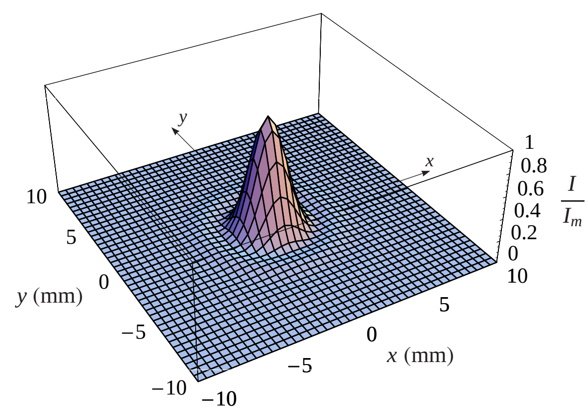

Figure 7.23

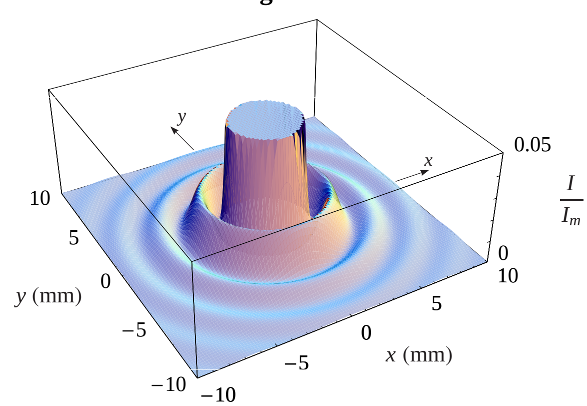

Figure 7.24

PROBLEMS

7.1 Suppose $\lambda = 632.8\,\text{nm}$, $A = 10^4\,\text{mV}$, $t = 0\,\text{s}$, and the phase constant is $\phi = 0\,\text{rad}$—these are the values used to construct the graph in Figure 7.2. (a) Calculate the angular wave number k in rad/nm. (b) Use Equation 7.1 to calculate E in mV/nm at $\rho = 1000\,\text{nm}$. (c) By inspecting the graph in Figure 7.2, we see that E has a minimum near $\rho = 1000\,\text{nm}$. Determine the values of ρ and E at this minimum.

7.2 Write Equation 7.9 for three pinholes and draw a diagram of this case patterned after Figure 7.5.

7.3 Show the algebraic steps to get Equation 7.21.

Note: In the problems that follow for diffraction produced by a rectangular aperture, use $\lambda = 632.8\,\text{nm}$, $D = 0.800\,\text{m}$, $2x_0 = 0.200\,\text{mm}$, and $2y_0 = 0.400\,\text{mm}$ whenever these values are needed.

7.4 Suppose $x = 0.500\,\text{mm}$ and $y = 0.100\,\text{mm}$, calculate (a) α and β, (b) u and v, and (c) I/I_m.

7.5 The solution $u = 0$ is a solution of $u - \tan u = 0$. But it must also be a solution of the $(u - \tan u)/u^2$ expression in Equation 7.44b. However, when we substitute $u = 0$ into this latter expression, we get the indeterminate value of $0/0$. Use l'Hospital's rule to show that $u = 0$ is indeed a solution.

7.6 Use the values for x_1, x_2, x_3 in Equation 7.57 to calculate I/I_m on the positive x axis for the first, second, and third-order maxima.

7.7 A problem similar to Problem 7.6 but use the values for y_1, y_2, y_3 in Equation 7.59 to calculate I/I_m on the positive y axis for the first, second, and third-order maxima.

7.8 One of the secondary maxima in a quadrant is listed as (x_1, y_1) in Equation 7.60. Use the values of x_1, y_1 in Equations 7.57 and 7.59, respectively, and calculate I/I_m at this point.

7.9 The values $x = 3.5, 3.6, 3.7\,\text{mm}$ are in the vicinity of the first order maximum on the positive x axis. Calculate I/I_m at these values.

7.10 (a) Find the value of x on the positive x axis that corresponds to the seventh-order minimum. (b) Find the value of y on the positive y axis that corresponds to the seventh-order minimum.

7.11 (a) Use the flow chart in Figure 7.18 to find the solution to $\tan u = u$ for the fourth order u_4 to a precision of 0.001. If you start with $u_{lo} = 13.8$ and $u_{hi} = 14.1$, then 13 loops are required. (b) State the value of u_4 in terms of $\pi/2$. (c) Find the corresponding value of x_4.

Note: In the problems that follow for diffraction produced by a circular aperture, use $\lambda = 632.8\,\text{nm}$, $D = 0.800\,\text{m}$, and $d = 0.500\,\text{mm}$ whenever these values are needed.

7.12 Evaluate the radii s of the first three circles that give the minima for I/I_m on the observation screen.

7.13 Draw a graph of I/I_m using Equation 7.74a for the interval $-8 \le v \le 8$. Use *Mathematica* or other program of your choice to draw the graph. If you use a pocket calculator, draw a rough graph using the values in the table of Figure 7.22.

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Chapter 7: Answers to Problems

7.1 (a) 0.009929 rad/nm; (b) −8.75 mV/nm; (c) 949 nm, −10.5 mV/nm

7.4 (a) 0.000625, 0.000125; (b) 0.621, 0.248; (c) 0.860

7.6 0.0472, 0.0165, 0.00834

7.7 0.0472, 0.0165, 0.00834

7.8 0.00223

7.9 0.0461, 0.0472, 0.0467

7.10 (a) 17.7 mm, (b) 8.86 mm

7.11 (a) 14.066, (b) 8.9547, (c) 11.333 mm

7.12 1.235, 2.261, 3.28 mm