Chapter 8—Outline

Fresnel Diffraction

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Chapter 8
Fresnel Diffraction

8.1 Introduction

In Chapter 7, we derived Equation 7.14

$$\mathbf{E}(P) = B \int_{S'} \frac{e^{i(k\rho - \omega t)}}{\rho} \, dS' \tag{8.1}$$

which we renumber here for convenience. This equation allows the evaluation of the electric field **E** at a point P on an observation screen for the case illustrated in Figure 7.7 under the following conditions: 1) the light falling on the aperture S' from the left is composed of parallel wavefronts of monochromatic, coherent light, and 2) the light is uniformly distributed over the aperture S' with the same phase (that is, the phase constant ϕ is zero). When we derived this equation before in Section 7.2, we mentioned that it was easier to separate the work into two parts: the first called Fraunhofer diffraction, the second called Fresnel diffraction. In Fraunhofer diffraction, the point P is on an observation screen far from the aperture. If we would imagine that the boundary of the aperture were projected forward to the observation screen, then the interesting variation in the intensity for Fraunhofer diffraction occurs outside this boundary: we investigated this kind of diffraction in Chapter 7. In Fresnel diffraction, the topic of this chapter, the point P is closer to the aperture, and the interesting behavior of the intensity occurs within the aperture projection on the observation screen.

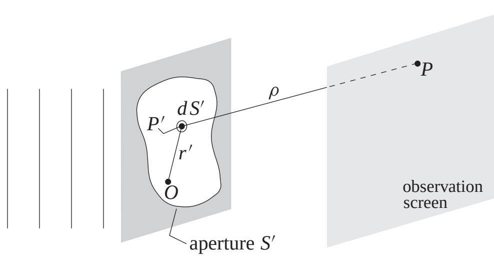

Figure 8.1

8.2 Fresnel Diffraction by a Single Aperture

We begin our work exactly like we did in Section 7.3.1 using the same diagram in Figure 7.8, except here we name it Figure 8.2. The derivation of the equations from Equation 7.15 to 7.20 holds for our current discussion, but we only need Equation 7.20; that is,

$$\rho^2 = \left(x - x'\right)^2 + \left(y - y'\right)^2 + D^2 \tag{8.2}$$

The difference between Fresnel diffraction and Fraunhofer diffraction begins with the way we treat this equation.

In Fraunhofer diffraction, we said that the terms x'^2 and y'^2 were small enough compared to other terms that they could be ignored. In Fresnel diffraction, we must leave these

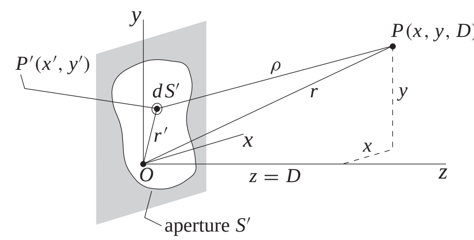

Figure 8.2

terms in the equation. We rewrite Equation 8.2 by factoring out D^2 to get

$$\rho^2 = D^2 \left[1 + \frac{\left(x - x'\right)^2}{D^2} + \frac{\left(y - y'\right)^2}{D^2} \right] \tag{8.3}$$

In Fresnel diffraction, $|x - x'|$ and $|y - y'|$ are small enough compared to D that we have

$$\left| \frac{x - x'}{D} \right| << 1 \qquad \text{and} \qquad \left| \frac{y - y'}{D} \right| << 1 \tag{8.4}$$

and then with the help of the binomial theorem

$$\begin{aligned} \rho &= D \left[1 + \frac{\left(x - x'\right)^2}{D^2} + \frac{\left(y - y'\right)^2}{D^2} \right]^{1/2} \\ &\approx D \left[1 + \frac{1}{2} \frac{\left(x - x'\right)^2}{D^2} + \frac{1}{2} \frac{\left(y - y'\right)^2}{D^2} \right] \\ &= D + \frac{\left(x - x'\right)^2}{2D} + \frac{\left(y - y'\right)^2}{2D} \end{aligned} \tag{8.5}$$

It is this expression in Equation 8.5 that we substitute into Equation 8.1. We see that ρ appears twice in Equation 8.1: in the complex exponential and in the denominator. In the complex exponential, ρ is composed of the D term, and the much smaller $(x - x')^2/(2D)$ and $(y - y')^2/(2D)$ terms—even these small terms are important in the complex exponential because when it is expanded into rectangular form by Euler's identity of Equation 5.11 it gives sine and cosine functions where the small terms are important. However, in the denominator the small terms that are part of ρ are no longer important, and we can replace it simply by D. Using these comments, we can substitute Equation 8.5 into Equation 8.1 to obtain

$$\mathbf{E}(P) = B\int_{S'} \frac{e^{i(k\rho-\omega t)}}{\rho}\,dS'$$

$$\approx B\int_{S'} \frac{e^{i\{k[D+(x-x')^2/(2D)+(y-y')^2/(2D)]-\omega t\}}}{D}\,dS'$$

$$= \frac{Be^{i(kD-\omega t)}}{D}\int_{S'} e^{ik(x'-x)^2/(2D)}\cdot e^{ik(y'-y)^2/(2D)}\,dS' \quad (8.6)$$

where in the last equation, we have interchanged the x and x' terms, and the y and y' terms. Because the expressions in which these terms arise are squared, it is certainly correct to perform this interchange; it will make the mathematical manipulations that follow a little cleaner by getting rid of some minus signs that would otherwise arise. We now specify the shape of the aperture.

8.3 Fresnel Diffraction by a Single Aperture that is Rectangular

8.3.1 Setting up the problem

We choose a rectangular aperture, as shown in Figure 8.3. We want to obtain the intensity I at the point P, which we regard as located on an observation screen a distance D from the aperture. As indicated in the diagram, the origin O does not have to be located at the midpoint of the rectangular aperture. To get the intensity I, we must first evaluate $\boldsymbol{E}(P)$ in Equation 8.6. To carry out this evaluation, we shall need two special functions called the Fresnel integrals; we list them now because it will make our derivation somewhat easier later:

$$C(u) = \int_0^u \cos\left(\frac{\pi}{2}u'^2\right)du' \quad (8.7a)$$

and

$$S(u) = \int_0^u \sin\left(\frac{\pi}{2}u'^2\right)du' \quad (8.7b)$$

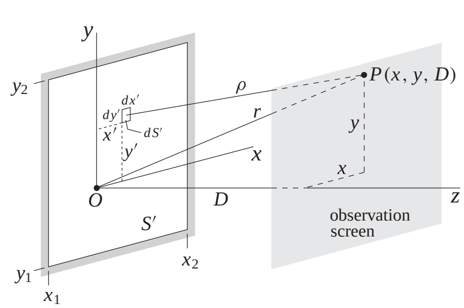

Figure 8.3

As indicated in Figure 8.3, we choose dS' to be rectangular and equal to $dx'\,dy'$; substituting this expression for dS' into Equation 8.6, and separating the integration into an x' part and a y' part, we obtain

$$\mathbf{E}(P) = \frac{Be^{i(kD-\omega t)}}{D}\left[\int_{x_1}^{x_2} e^{ik(x'-x)^2/(2D)}\,dx'\right]\cdot\left[\int_{y_1}^{y_2} e^{ik(y'-y)^2/(2D)}\,dy'\right] \quad (8.8)$$

Next we replace the wave number k by $2\pi/\lambda$, and rearrange the exponentials slightly:

$$\mathbf{E}(P) = \frac{Be^{i(kD-\omega t)}}{D}\left[\int_{x_1}^{x_2} e^{i\frac{\pi}{2}\frac{2}{\lambda D}(x'-x)^2}\,dx'\right]\cdot\left[\int_{y_1}^{y_2} e^{i\frac{\pi}{2}\frac{2}{\lambda D}(y'-y)^2}\,dy'\right] \quad (8.9)$$

We now change the variable of integration by defining

$$u' = \sqrt{\frac{2}{\lambda D}}\,(x'-x), \qquad v' = \sqrt{\frac{2}{\lambda D}}\,(y'-y) \quad (8.10)$$

which makes Equation 8.9 become

$$\mathbf{E}(P) = \frac{\lambda}{2}\,Be^{i(kD-\omega t)}\left[\int_{u_1}^{u_2} e^{i\frac{\pi}{2}u'^2}\,du'\right]\cdot\left[\int_{v_1}^{v_2} e^{i\frac{\pi}{2}v'^2}\,dv'\right] \quad (8.11)$$

where

$$u_1 = \sqrt{\frac{2}{\lambda D}}\,(x_1-x), \qquad u_2 = \sqrt{\frac{2}{\lambda D}}\,(x_2-x) \quad (8.12a)$$

$$v_1 = \sqrt{\frac{2}{\lambda D}}\,(y_1-y), \qquad v_2 = \sqrt{\frac{2}{\lambda D}}\,(y_2-y) \quad (8.12b)$$

We take the first integral inside the brackets in Equation 8.9, and manipulate it, using the rules of calculus and Euler's identity of Equation 5.11, into a form that will allow the use of the Fresnel integrals in Equations 8.7. The first step is to write

$$\int_{u_1}^{u_2} e^{i\frac{\pi}{2}u'^2}\,du' = \int_0^{u_2} e^{i\frac{\pi}{2}u'^2}\,du' - \int_0^{u_1} e^{i\frac{\pi}{2}u'^2}\,du' \quad (8.13a)$$

The two integrals on the right of the equals sign in Equation 8.13a are called complex Fresnel integrals. We separate these integrals into real and imaginary parts by using Euler's identity (see Equation 5.11). For the first integral to the right of the equals sign we have

$$\int_0^{u_2} e^{i\frac{\pi}{2}u'^2}\,du' = \int_0^{u_2} \cos\left(\frac{\pi}{2}u'^2\right) du' + i\int_0^{u_2} \sin\left(\frac{\pi}{2}u'^2\right) du' = C(u_2) + iS(u_2) \tag{8.13b}$$

where in the last step we substitute the Fresnel equations as defined in Equations 8.7. Similary, for the second integral to the right of the equals sign in Equation 8.13a, we have

$$\int_0^{u_1} e^{i\frac{\pi}{2}u'^2}\,du' = C(u_1) + iS(u_1) \tag{8.13c}$$

Finally, we substitute Equations 8.13b and 8.13c into Equation 8.13a, and obtain

$$\int_{u_1}^{u_2} e^{i\frac{\pi}{2}u'^2}\,du' = [C(u_2) + iS(u_2)] - [C(u_1) + iS(u_1)] = [C(u_2) - C(u_1)] + i\,[S(u_2) - S(u_1)] \tag{8.14}$$

and we have obtained the result in terms of Fresnel integrals for the u integral between the brackets in Equation 8.11. Similary, for the v integral between the brackets in Equation 8.11, we get

$$\int_{v_1}^{v_2} e^{i\frac{\pi}{2}v'^2}\,dv' = [C(v_2) + iS(v_2)] - [C(v_1) + iS(v_1)] = [C(v_2) - C(v_1)] + i\,[S(v_2) - S(v_1)] \tag{8.15}$$

Substituting Equations 8.14 and 8.15 into Equation 8.11, we obtain the expression for $\mathbf{E}(P)$:

$$\mathbf{E}(P) = \frac{\lambda B}{2} e^{i(kD-\omega t)} \{[C(u_2) - C(u_1)] + i\,[S(u_2) - S(u_1)]\} \cdot \{[C(v_2) - C(v_1)] + i\,[S(v_2) - S(v_1)]\} \tag{8.16}$$

As usual, the intensity I at the point P is found by using Equation 5.85, which says to take **E**(P) in Equation 8.16, multiply by its complex conjugate, and then multiply by the constant C:

$$I(P) = C\,\mathbf{E}(P)\,\mathbf{E}^\star(P) = C\left(\frac{\lambda B}{2}\right)^2 \left\{[C(u_2) - C(u_1)]^2 + [S(u_2) - S(u_1)]^2\right\} \cdot \left\{[C(v_2) - C(v_1)]^2 + [S(v_2) - S(v_1)]^2\right\} \tag{8.17}$$

We now investigate the properties of the Fresnel integrals C and S in more detail (see Equations 8.7).

8.3.2 The Fresnel integrals and the Cornu spiral

The Fresnel integrals are named after Augustin Jean Fresnel, a French physicist who worked for the government as a civil engineer; he started his work on the wave theory of light a few years after 1800. As a child, Fresnel developed slowly, not learning to read until the rather late age of eight, and died at the early age of 39. But his scientific skills developed rapidly. After becoming interested in the wave theory of light, he began to construct a mathematical basis for the wave theory. Huygens and others had believed that light waves were longitudinal waves. Their reasoning was that waves—such as sound waves, water waves, waves traveling down a metal bar—need a medium in which to travel. Since light traveled through space, along with the earth, moon, and other planetary and stellar objects, the medium for light should be virtually massless, like air; only longitudinal waves can be transmitted in such media. Acting on a suggestion that light waves might be transverse, Fresnel built up a theory that was able to explain the double refraction of Iceland spar; workers had been unable to do this with the longitudinal wave theory. A description of the properties of polarized light was also possible. However, fellow physicists had trouble accepting light waves as transverse, because such waves normally travel in a solid medium. And because of the high speed of light, this solid medium had to be very rigid. Thus, how could such objects as the earth, moon, and the planets move so easily through it. Nevertheless, because the transverse wave theory described the many properties of light so well, scientists began to accept it. The mysterious medium that supported the travel of transverse light waves was called the ether, coming from a Greek word meaning "upper air." The search for the ether continued for many years, it was unsuccessful, and finally culminated in the formation of the special theory of relativity. Light, it was eventually concluded, had many very strange properties—an description of these properties has played an important role in the development of physics.

The Fresnel integrals vary roughly like the sine function, except that they are always positive for a positive argument, say u. For convenience of working with these functions, we list them again:

$$C(u) = \int_0^u \cos\left(\frac{\pi}{2} u'^2\right) du' \tag{8.18a}$$

$$S(u) = \int_0^u \sin\left(\frac{\pi}{2} u'^2\right) du' \tag{8.18b}$$

We can look up values of these functions in tables, or use a program like *Mathematica* to evaluate them. We graph these functions in Figure 8.4, and list some values of $C(u)$ and $S(u)$ in the table of Figure 8.5.

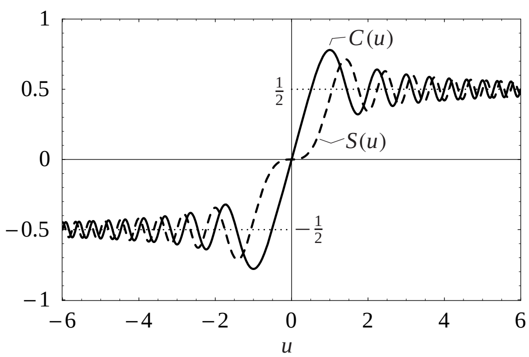

Figure 8.4

u	$C(u)$	$S(u)$
0.	0.	0.
0.5	0.492344	0.064732
1.	0.779893	0.438259
1.5	0.445261	0.697505
2.	0.488253	0.343416
2.5	0.457413	0.619182
3.	0.605721	0.496313
3.5	0.532572	0.415248
4.	0.498426	0.420516
4.5	0.526026	0.434273
5.	0.563631	0.499191
5.5	0.478421	0.553684
6.	0.499531	0.44696
6.5	0.481603	0.545376
7.	0.545467	0.499705
7.5	0.516018	0.460701
8.	0.499802	0.460214

Figure 8.5

Inspecting the graph and the table suggests that as u gets large both $C(u)$ and $S(u)$ approach the value of 1/2. This result can be shown to be true, but would take too long to do it here; thus, we simply state

$$C(\infty) = S(\infty) = \frac{1}{2} \tag{8.19}$$

A second result, and this one can be shown quite easily with the rules of calculus, is that both $C(u)$ and $S(u)$ are odd functions:

$$C(-u) = -C(u) \qquad \text{and} \qquad S(-u) = -S(u) \tag{8.20}$$

A useful graphical interpretation can be given to the Fresnel integrals that helps to interpret the results obtained when they are applied to an aperture. As a first step, we need to put $C(u)$ and $S(u)$ together to obtain the complex Fresnel integral $F(u)$. We use Equation 8.13b as a guide, but work backwards:

$$\begin{aligned} C(u) + iS(u) &= \int_0^u \cos\left(\frac{\pi}{2}u'^2\right) du' \\ &\qquad + i\int_0^u \sin\left(\frac{\pi}{2}u'^2\right) du' \\ &= \int_0^u \left[\cos\left(\frac{\pi}{2}u'^2\right) + i\sin\left(\frac{\pi}{2}u'^2\right)\right] du' \\ &= \int_0^u e^{i\frac{\pi}{2}u'^2}\, du' \\ &= F(u) \end{aligned} \tag{8.21}$$

Let's assume that u is positive, and break the interval from 0 to u over which the integration is taking place into N segments of equal length

$$\Delta u' = \frac{u}{N} \tag{8.22}$$

The value of u' at the beginning of each of these segments is

$$u' = 0,\ \Delta u',\ 2\,\Delta u',\ 3\,\Delta u',\ \ldots,\ (N-1)\Delta u'$$

or more briefly

$$u'_j = j\,\Delta u' \qquad j = 0, 1, 2, \ldots, N-1 \tag{8.23}$$

The complex Fresnel integral in Equation 8.21 can then be approximated as a summation:

$$F(u) \approx \sum_{j=0}^{N-1} e^{i\frac{\pi}{2}u_j'^2}\,\Delta u' \tag{8.24}$$

To see how this approach works, let's take $N = 3$ so that $\Delta u' = u/3$. Then Equation 8.23 gives the u' values at the beginning of each straight-line segment (or as we shall see, at the beginning of each phasor)

$$u'_0 = 0, \qquad u'_1 = \Delta u', \qquad u'_2 = 2\Delta u' \tag{8.25}$$

and Equation 8.24 yields

$$\begin{aligned} F(u) &\approx e^{i\frac{\pi}{2}{u'_0}^2}\,\Delta u' + e^{i\frac{\pi}{2}{u'_1}^2}\,\Delta u' + e^{i\frac{\pi}{2}{u'_2}^2}\,\Delta u' \\ &= \Delta u'\, e^{i\theta'_0} + \Delta u'\, e^{i\theta'_1} + \Delta u'\, e^{i\theta'_2} \end{aligned} \tag{8.26}$$

where

$$\theta'_0 = \frac{\pi}{2}{u'_0}^2, \qquad \theta'_1 = \frac{\pi}{2}{u'_1}^2, \qquad \theta'_2 = \frac{\pi}{2}{u'_2}^2 \tag{8.27a}$$

or, substituting the values in Equation 8.25:

$$\theta'_0 = 0, \qquad \theta'_1 = \frac{\pi}{2}\Delta u'^2, \qquad \theta'_2 = \frac{\pi}{2}\,4\Delta u'^2 \tag{8.28b}$$

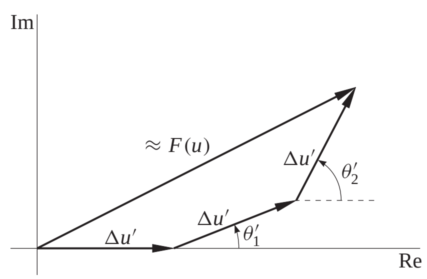

Figure 8.6(a)

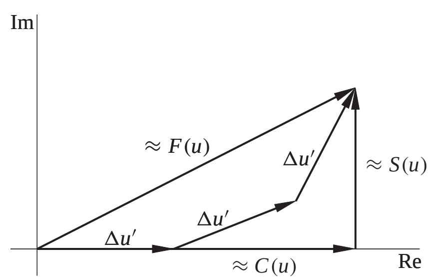

Figure 8.6(b)

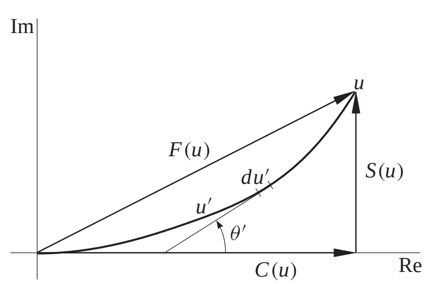

Figure 8.6(c)

Looking at Equation 8.26, we see that each of the terms is a phasor in polar form with the same magnitude of $\Delta u'$, but with different phase angles: the angles are $\theta'_0 = 0$, θ'_1, and θ'_2, as given in Equation 8.28b. We represent each of the phasors in the complex plane, as shown in Figure 8.6(a), and draw them in head-to-tail form because phasors add like vectors; in fact, we see that Equation 8.26 says that the phasor sum approximates $F(u)$. As we mentioned at the top of the previous column, it is important to note that the $u'_0 = 0$ value occurs at the tail of the first phaser, the u'_1 value at the tail of the second phasor, and u'_2 at the tail of the third. Also, if we add all the magnitudes $\Delta u'$, we get u; that is, $3\Delta u' = u$, since, remember, we took $N = 3$ so that $\Delta u' = u/3$ (see Equation 8.22). We can also look at Figure 8.6(a) and see that there are three $\Delta u'$s, each with the length of $u/3$, and so the sum is u. Thus, we can say the phasors form a section of a polygon along which the u' values can be placed and it has a perimeter equal to u.

Looking back at Equation 8.21, we see that $F(u)$ is expressed in rectangular form as $C(u) + iS(u)$; thus, the phasor that approximates $F(u)$ in the complex plane is regarded as the phasor sum of the rectangular components that approximate $C(u)$ and $S(u)$, as shown in Figure 8.6(b). Thus, the real axis (the horizontal axis) represents $C(u)$, and the imaginary axis (the vertical axis) represents $S(u)$; this correspondence is very important in constructing the Cornu spiral.

Now let's choose a larger N, in fact, let N approach infinity. Then the number of phasors grows larger and larger, and their magnitudes get smaller and smaller so that each has an infinitesimal magnitude of du'. The sum of all these du's still equals u, and the section of the polygon we had before now tends to an arc of length u, as shown in Figure 8.6(c), where we have not drawn the du' phasor arrowheads for simplicity. Also, the u' values are measured along this arc, and the tangent line drawn for an arbitrary infinitesimal phasor (see Equation 8.26)

$$du'\, e^{i\theta'}$$

at some particular u' value makes a phase angle (see Equation 8.27a) of $\theta' = \frac{\pi}{2}u'^2$ with the horizontal, as shown in Figure 8.6(c). The phasor sum now equals, not just approximates, the complex Fresnel integral $F(u)$ and has the components $C(u)$ and $S(u)$. The arc of length u is a section of a curve called the Cornu spiral.

To obtain a more complete Cornu spiral, all we have to do is graph $S(u)$ versus $C(u)$; the graph of the Cornu spiral when u varies between -4 and 4 is shown in Figure 8.7. Since both functions $C(u)$ and $S(u)$ approach the value $1/2$ as u approaches infinity, and the value $-1/2$ as u approaches minus infinity, the Cornu spiral winds around points we call z and $-z$: the point z represents the point $(1/2, 1/2)$, and similarly, $-z$ represents the point $(-1/2, -1/2)$. The curve approaches these points asymptotically. It is important to remember that $S(u)$ is still graphed along the imaginary axis, and $C(u)$ along the real axis, just as we did before in the diagrams of Figure 8.6.

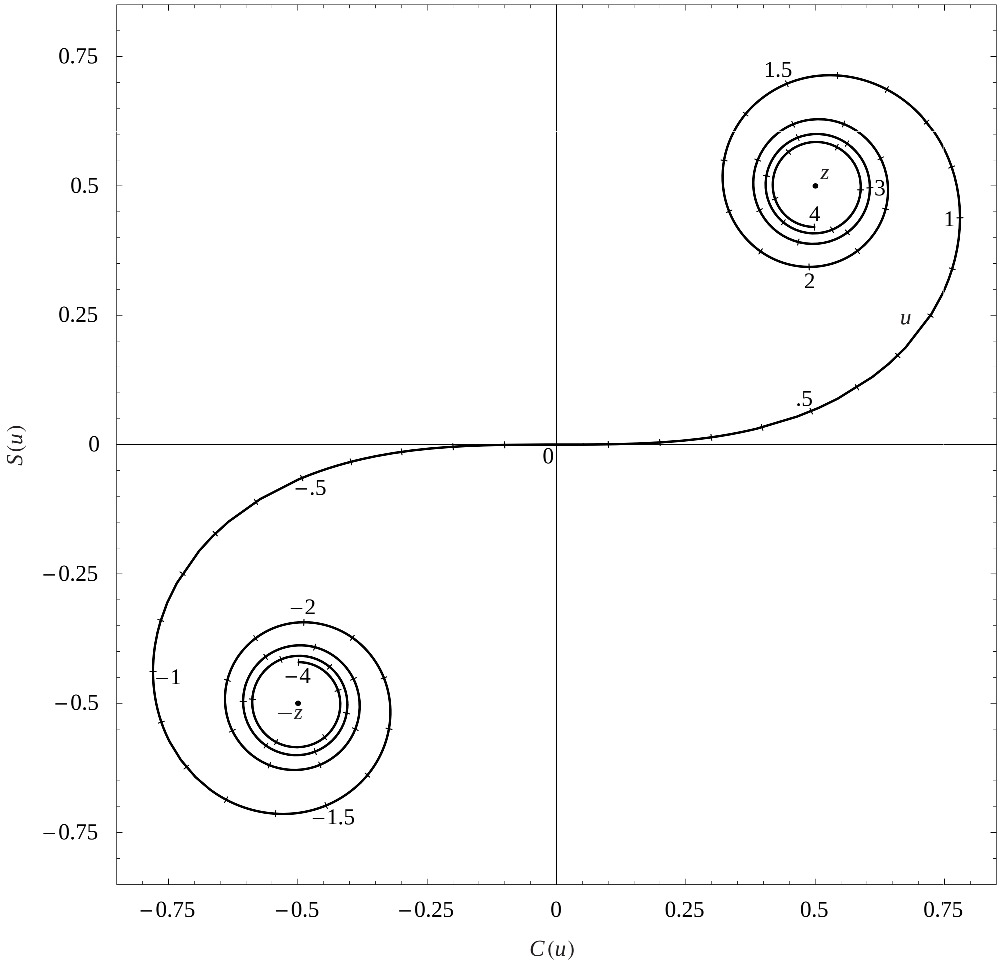

Figure 8.7

The real usefulness of the Cornu spiral is to give a graphical meaning to the evaluation of $F(u)$, the complex Fresnel integral, which we listed before as Equation 8.21; given in abbreviated form it is:

$$F(u) = C(u) + iS(u) \tag{8.29}$$

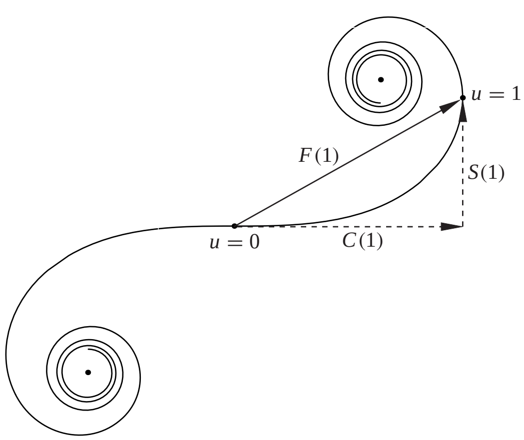

Figure 8.8

It is beneficial to look back at Equation 8.21 to understand the rather “messy” integrals that $F(u)$, $C(u)$, and $S(u)$ represent. Suppose we want to evaluate $F(1)$, that is, we want $F(1) = C(1) + iS(1)$. All we need to do is draw a phasor from the origin at $u = 0$ to $u = 1$ on the graph of the Cornu spiral, as we show in Figure 8.8. The horizontal component of $F(1)$ represents $C(1)$ and the vertical component represents $S(1)$. As we shall see shortly, we only need the magnitude of $F(1)$, which we calculate with the help of the complex conjugate $F(1)^\star$:

$$\begin{aligned} |F(1)| &= \text{magnitude of } F(1) = \sqrt{F(1) \cdot F(1)^\star} \\ &= \sqrt{[C(1) + iS(1)] \cdot [C(1) - iS(1)]} \\ &= \sqrt{C(1)^2 + S(1)^2} \end{aligned} \tag{8.30}$$

Since the complex Fresnel integral $F(u)$ is a phasor on the diagram of a Cornu spiral, and since phasors add and subtract as vectors, we can make a convenient representation of several $F(u)$ phasors and graphically see how they relate to each other, as we show in Figure 8.9. The phasor $F(u_1)$ is drawn from the origin to the u_1 point (assumed to be negative) on the Cornu spiral, and the same is done for the $F(u_2)$ phasor. Then since the tails of the $F(u_1)$ and $F(u_2)$ phasors touch, it is easy to see that the difference phasor $F(u_2) - F(u_1)$ is drawn from the head of the $F(u_1)$ phasor to the head of the $F(u_2)$ phasor (see Section 5.2.4). Difference phasors of this kind are very important in the evaluation of the Fresnel integrals. For this diagram, remembering how $F(u)$ is related to $C(u)$ and $S(u)$, we have

$$\begin{aligned} F(u_2) - F(u_1) &= [C(u_2) + iS(u_2)] - [C(u_1) - iS(u_1)] \\ &= C(u_2) - C(u_1) + i[S(u_2) - S(u_1)] \end{aligned}$$

and then

$$\begin{aligned} &|F(u_2) - F(u_1)| \\ &\quad = \text{magnitude of } [F(u_2) - F(u_1)] \\ &\quad = \sqrt{[F(u_2) - F(u_1)] \cdot [F(u_2) - F(u_1)]^{\star}} \\ &\quad = \sqrt{[C(u_2) - C(u_1)]^2 + [S(u_2) - S(u_1)]^2} \qquad (8.31) \end{aligned}$$

Ultimately, we want the intensity $I(P)$. The equation for its calculation is given by Equation 8.17, and we see that the u part of the expression for $I(P)$ is

$$[C(u_2) - C(u_1)]^2 + [S(u_2) - S(u_1)]^2$$

which we get by squaring Equation 8.31, namely,

$$\begin{aligned} &|F(u_2) - F(u_1)|^2 \\ &\quad = [C(u_2) - C(u_1)]^2 + [S(u_2) - S(u_1)]^2 \qquad (8.32) \end{aligned}$$

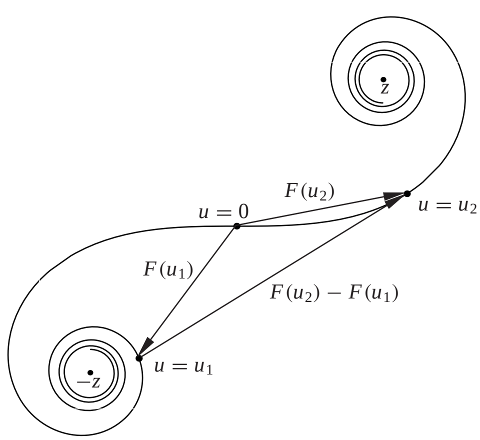

Figure 8.9

Thus, in general, the square of the magnitude $F(u_2) - F(u_1)$ that we can read off a Cornu spiral gives a geometrical procedure to obtain the u part of the intensity $I(P)$. For the v part of Equation 8.17, we would simply imagine another Cornu spiral drawn in terms of v instead of u. It is in this way that we can use the Cornu spiral to graphically represent the behavior of the intensity $I(P)$ in Equation 8.17.

It is also easy to see how expressions like Equations 8.31 and 8.32 might vary as the position of u_2 moves along the Cornu spiral; that is, as the head of $F(u_2)$ moves along the Cornu spiral toward the positive endpoint z. We see that the magnitude of $F(u_2) - F(u_1)$ becomes greatest when the head reaches the top of the spiral (by inspection of Figure 8.7, near $u_2 = 1.4$ or so), and then goes through many maxima and minima as u_2 winds around and around the spiral. The same thing can be said about the tail of $F(u_2) - F(u_1)$ as the u_1 point moves toward negative endpoint $-z$.

8.3.3 The intensity I_0

Let's review what we are trying to do. Our goal is to describe the intensity $I(P)$ after parallel wavefronts of monochromatic, coherent light have passed through a rectangular aperture; we call the intensity of these incident parallel wavefronts I_0. The rectangular aperture that we working with is shown again in Figure 8.10; it is a redraw of Figure 8.3. We showed

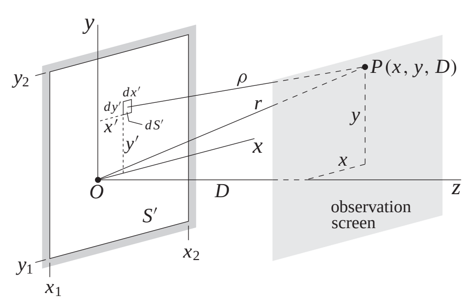

Figure 8.10

in Section 8.3.1 that the intensity at the point P is given by Equation 8.17 with the expressions for the us and vs given in Equations 8.12. We list these equations again:

$$\begin{aligned} I(P) = C\left(\frac{\lambda B}{2}\right)^2 &\left\{[C(u_2) - C(u_1)]^2 \right. \\ &\left. + [S(u_2) - S(u_1)]^2\right\} \\ &\cdot \left\{[C(v_2) - C(v_1)]^2 \right. \\ &\left. + [S(v_2) - S(v_1)]^2\right\} \qquad (8.33) \end{aligned}$$

where

$$u_1 = \sqrt{\frac{2}{\lambda D}}\,(x_1 - x), \qquad u_2 = \sqrt{\frac{2}{\lambda D}}\,(x_2 - x) \quad (8.34a)$$

$$v_1 = \sqrt{\frac{2}{\lambda D}}\,(y_1 - y), \qquad v_2 = \sqrt{\frac{2}{\lambda D}}\,(y_2 - y) \quad (8.34b)$$

We now want to write $I(P)$ as given in Equation 8.33 in a simplified form by writing it in terms of I_0 so that we can remove the $C(\lambda B/2)^2$ term. This procedure is different than the one we used in the previous chapter with Fraunhofer diffraction, where we expressed $I(P)$ in terms of I_m, the maximum value of $I(P)$. In Fresnel diffraction, it is easier to express $I(P)$ in terms of I_0 because a Fresnel diffraction pattern frequently does not have a clearly defined maximum.

If there is no aperture, then the intensity $I(P)$ should equal I_0, the intensity of the incident radiation. Referring to Figure 8.10, we can create the "no aperture" condition by allowing x_2 and y_2 to go to plus infinity, and x_1 and y_1 to minus infinity—or at least create conditions so that in Equations 8.34 we can say that $u_1 = v_1 \approx -\infty$ and $u_2 = v_2 \approx \infty$. Then remembering that $C(\infty) = S(\infty) = 1/2$ and that also $C(-\infty) = S(-\infty) = -1/2$ (see Equations 8.19 and 8.20), we have

$$
\begin{aligned}
I(P) = I_0 = C\left(\frac{\lambda B}{2}\right)^2 &\left\{[C(\infty) - C(-\infty)]^2 + [S(\infty) - S(-\infty)]^2\right\} \\
&\cdot \left\{[C(\infty) - C(-\infty)]^2 + [S(\infty) - S(-\infty)]^2\right\} \\
= C\left(\frac{\lambda B}{2}\right)^2 &\left\{\left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right]^2 + \left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right]^2\right\} \\
&\cdot \left\{\left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right]^2 + \left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right]^2\right\} \\
= C\left(\frac{\lambda B}{2}\right)^2 &\{4\} \qquad (8.35)
\end{aligned}
$$

which gives

$$C\left(\frac{\lambda B}{2}\right)^2 = \frac{I_0}{4} \qquad (8.36)$$

so that Equation 8.33 can now be written more simply in terms of I_0 as

$$
\begin{aligned}
I(P) = \frac{I_0}{4} &\left\{[C(u_2) - C(u_1)]^2 + [S(u_2) - S(u_1)]^2\right\} \\
&\cdot \left\{[C(v_2) - C(v_1)]^2 + [S(v_2) - S(v_1)]^2\right\} \qquad (8.37)
\end{aligned}
$$

8.3.4 A single edge

Fresnel diffraction occurs at a single edge. We make a single edge out of the rectangular aperture shown in Figure 8.10 by allowing x_2 and y_2 to go to plus infinity, and y_1 to minus infinity; the edge obtained is shown in Figure 8.11. Substituting

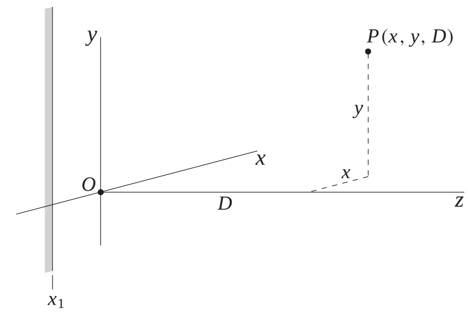

Figure 8.11

these infinity values into Equations 8.34 gives $v_1 = -\infty$ and $u_2 = v_2 = \infty$. Then, recalling that $C(\infty) = S(\infty) = 1/2$ and also $C(-\infty) = S(-\infty) = -1/2$ (see Equations 8.19 and 8.20), Equation 8.37 gives

$$
\begin{aligned}
I(P) &= \frac{I_0}{4}\left\{[C(\infty) - C(u_1)]^2 + [S(\infty) - S(u_1)]^2\right\} \\
&\quad \cdot \left\{[C(\infty) - C(-\infty)]^2 + [S(\infty) - S(-\infty)]^2\right\} \\
&= \frac{I_0}{4}\left\{\left[\frac{1}{2} - C(u_1)\right]^2 + \left[\frac{1}{2} - S(u_1)\right]^2\right\} \\
&\quad \cdot \left\{\left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right]^2 + \left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right]^2\right\} \\
&= \frac{I_0}{2}\left\{\left[\frac{1}{2} - C(u_1)\right]^2 + \left[\frac{1}{2} - S(u_1)\right]^2\right\} \qquad (8.38)
\end{aligned}
$$

where (see Equation 8.34a)

$$u_1 = \sqrt{\frac{2}{\lambda D}}\,(x_1 - x) \qquad (8.39)$$

Let us now divide through by I_0 in Equation 8.38, and graph $I(P)/I_0$ as a function of x. If we choose $\lambda = 632.8$ nm, $D = 160$ mm, and $x_1 = -0.5$ mm, we get the graph shown in Figure 8.12. The dashed line at $x = -0.5$ mm equals the value of x_1 and marks the position of the single edge. To understand what the graph means, it helps to look at Figure 8.11, and to keep in mind the above Equations 8.38 and 8.39. The values of x smaller than $x_1 = -0.5$ mm are negative values, where we see from Equation 8.39 that u_1 has its positive values, and represent positions behind the single edge; even behind the edge we see from the graph that there is some light, because I/I_0 is building up from zero. At $x = x_1 = -0.5$ mm, we are right at the edge and $u_1 = 0$.

As x becomes greater than $x_1 = -0.5$ mm, we move to the right (in the $+x$ direction) of the single edge in Figure 8.11, and u_1 becomes more and more negative, eventually approaching minus infinity. As x moves through these latter values, the graph in Figure 8.12 shows that I/I_0 oscillates about one, approaching one asymptotically as x goes to plus infinity; in other words, I oscillates about I_0, approaching I_0 asymptotically as x goes to plus infinity.

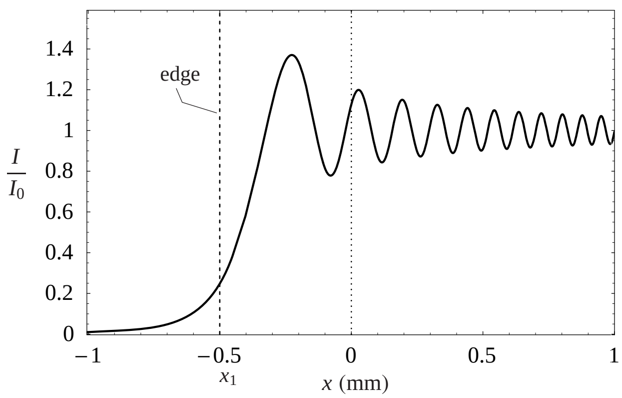

Figure 8.12

To see how the Cornu spiral is used to describe single-edge diffraction, we rewrite Equation 8.38 as

$$\begin{aligned}\frac{I(P)}{I_0} &= \frac{1}{2}\left\{\left[\frac{1}{2} - C(u_1)\right]^2 + \left[\frac{1}{2} - S(u_1)\right]^2\right\} \\ &= \frac{1}{2}\left\{[C(\infty) - C(u_1)]^2 + [S(\infty) - S(u_1)]^2\right\} \\ &= \frac{1}{2}\left\{[C(u_2) - C(u_1)]^2 + [S(u_2) - S(u_1)]^2\right\} \qquad (8.40)\end{aligned}$$

where (see Equation 8.34a)

$$u_1 = \sqrt{\frac{2}{\lambda D}}\,(x_1 - x), \qquad u_2 = \infty \qquad (8.41)$$

Then we recall Equation 8.32, which we obtained when we were discussing the meaning of phasors on the Cornu spiral; we multiply both sides of Equation 8.32 by 1/2 to make it easier to describe single-edge diffraction:

$$\begin{aligned}\frac{1}{2}|F(u_2) - F(u_1)|^2 &= \frac{1}{2}\left\{[C(u_2) - C(u_1)]^2 + [S(u_2) - S(u_1)]^2\right\} \qquad (8.42)\end{aligned}$$

We note how the right sides of Equations 8.40 and 8.42 correspond; thus, the $(1/2)|F(u_2) - F(u_1)|^2$ will help us understand the graph in Figure 8.12 from the point-of-view of the Cornu spiral—the squaring operation and the multiplying by one-half has to be performed mentally or in some other way (such as with a calculator).

To use the Cornu spiral, we draw three phasors on the spiral, as shown in Figure 8.13; each of the phasors is equal to $F(u_2) - F(u_1)$; and as Equation 8.42 shows, it is the magnitude of these phasors that is of interest. The heads of the phasors touch the point z because $u_2 = \infty$, a condition that results from $x_2 = \infty$. Except for the middle (or second phasor), the tails can move because their position is given by u_1, and the u_1 value depends on the value of x, which can change.

First, let's discuss the phasor with the tail described by u_1 = positive value; its positive value results from $x < x_1$ in Equation 8.41; that is, when we are behind the single edge. As x gets more and more negative (that is, we move farther and farther behind the edge), u_1 gets more positive so that the tail begins to move around the spiral in Figure 8.13 towards the point z; this movement makes the magnitude of the phasor get smaller in a uniform way which is consistent with the behavior of the graph in Figure 8.12 when $x < x_1$, where remember $x_1 = -0.5$ mm.

The second phasor has the tail value of $u_1 = 0$ because $x = x_1$, and the one-half times the square of the magnitude of this phasor equals the value of $I(P)/I_0$ in Equation 8.40; it represents what happens at a point on the edge in Figure 8.11, and equals I/I_0 at the point where the curve crosses the dashed line in Figure 8.12.

The third phasor has the tail value u_1 = negative value in Figure 8.13; these negative values for u_1 are given by Equation 8.41 when $x > x_1$, and correspond to the points x to the right of the edge in Figure 8.11. As x becomes greater than x_1, the values of u_1 become more negative and the tail of the phasor moves around the spiral towards the point $-z$ in Figure 8.13; this behavior means that the magnitude (and also one-half times the square of this magnitude) of this third phasor goes through a series of bigger and smaller values just as the graph does in Figure 8.12 for $x > x_1 = -0.5$ mm.

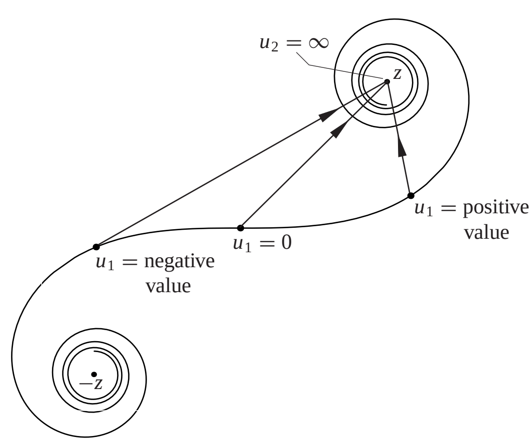

Figure 8.13

8.3.5 A slit

Suppose the monochromatic, coherent light travels through a slit. We make a slit out of the rectangular aperture of Figure 8.10 by setting y_1 to $-\infty$ and y_2 to ∞ to obtain the diagram shown in Figure 8.14.

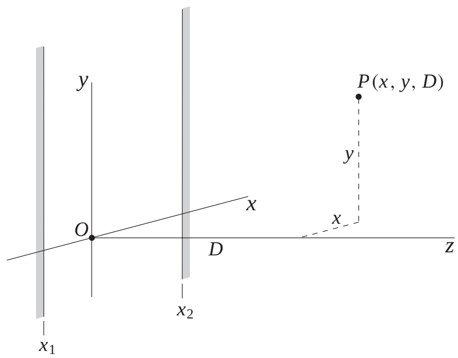

Figure 8.14

We follow the same procedure we used for the single edge, and substitute these infinity values into Equation 8.34b to get $v_1 = -\infty$ and $v_2 = \infty$. Then, just as before, we recall that $C(\infty) = S(\infty) = 1/2$, and correspondingly, that $C(-\infty) = S(-\infty) = -1/2$ (see Equations 8.19 and 8.20); thus, Equation 8.37 gives

$$\begin{aligned}\frac{I(P)}{I_0} &= \frac{1}{4}\left\{[C(u_2) - C(u_1)]^2 + [S(u_2) - S(u_1)]^2\right\} \\ &\quad \cdot \left\{[C(\infty) - C(-\infty)]^2 + [S(\infty) - S(-\infty)]^2\right\} \\ &= \frac{1}{2}\left\{[C(u_2) - C(u_1)]^2 + [S(u_2) - S(u_1)]^2\right\} \qquad (8.43)\end{aligned}$$

where

$$u_1 = \sqrt{\frac{2}{\lambda D}}\,(x_1 - x), \qquad u_2 = \sqrt{\frac{2}{\lambda D}}\,(x_2 - x) \qquad (8.44)$$

We now graph $I(P)/I_0$ in Equation 8.43 as a function of x, where we remember u_1 and u_2 are both functions of x as given by Equation 8.44. Suppose we again choose $\lambda = 632.8$ nm with $x_1 = -0.5$ mm and $x_2 = 0.5$ mm so that the width Δx of the slit is 1 mm. Because there is so much variation with the value of D, we draw graphs for several different values in Figure 8.15.

When D is rather large compared to Δx (for example, the $D = 790$ mm graph in Figure 8.15), the plot of I/I_0 approximates the behavior of Fraunhofer diffraction which we discussed in the previous chapter. But as D is made smaller, we note that the pattern becomes more complex with lateral dimensions that approximate the width of the slit; this latter property is quite different from what happened in Fraunhofer diffraction where the pattern could extend well beyond the width of the aperture. In Figure 8.15, the edges of the slit are marked by the dashed lines on either side of $x = 0$; namely, where $x = x_1 = -0.5$ mm and $x = x_2 = 0.5$ mm.

Just like we did for a single edge, we use the Cornu spiral to describe Fresnel diffraction for a slit. That is, we describe the behavior of Equation 8.43 by looking at Equation 8.42, which we display again:

$$\begin{aligned}&\frac{1}{2}|F(u_2) - F(u_1)|^2 \\ &\quad = \frac{1}{2}\left\{[C(u_2) - C(u_1)]^2 + [S(u_2) - S(u_1)]^2\right\} \qquad (8.45)\end{aligned}$$

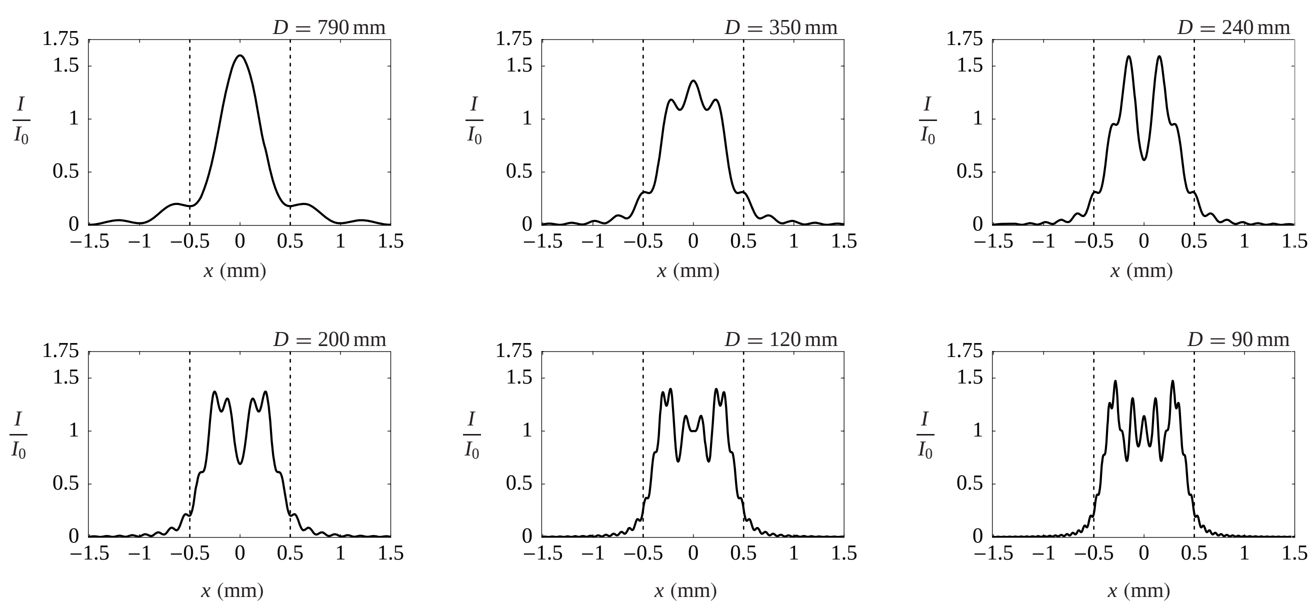

Figure 8.15

The big difference this time is that we can obtain a relationship involving $u_2 - u_1$. From Equation 8.44, we obtain

$$\Delta u = u_2 - u_1 = \sqrt{\frac{2}{\lambda D}}\,(x_2 - x_1) = \sqrt{\frac{2}{\lambda D}}\,\Delta x \quad (8.46)$$

which means that the length measured along the spiral curve (the arc length) has a fixed value between the head and tail of the phasor representing $F(u_2) - F(u_1)$ for the given values of λ, D, and Δx (remember that u is measured along the arc, or curve, making up the spiral).

For example, if we take $x_1 = -x_2$, as we assumed in Figure 8.15, then when $x = 0$, the phasor $F(u_2) - F(u_1)$is centered about $u = 0$, as shown in Figure 8.16. This centered feature follows from Equation 8.44: when $x = 0$, we have $u_1 = \sqrt{2/(\lambda D)}\,x_1$ and $u_2 = \sqrt{2/(\lambda D)}\,x_2$. Then, because $x_2 = -x_1$, we must have $u_2 = -u_1$. What is really important in Figure 8.16, is the arc length Δu measured between u_1 and u_2; this arc length must remain the same as the phasor moves with changing x (as long as we don't change λ, D, or Δx). If x increases through positive values, then Equation 8.44 shows that u_1 becomes more negative and u_2 will move through smaller positive values into negative values that continue to get more negative; the phasor $F(u_2') - F(u_1')$ illustrates a case where x has increased enough to move the phasor to the place shown. Even though the magnitude of $F(u_2') - F(u_1')$ is much smaller, the arc length between u_1' and u_2' is the same as before.

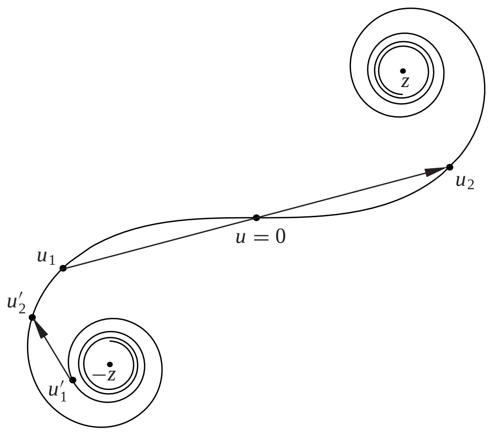

Figure 8.16

To obtain any of the graphs for the portion where $x \geq 0$ in Figure 7.14, we picture the phasor $F(u_2) - F(u_1)$ sliding down along the Cornu spiral toward $-z$ keeping the arc length constant between u_2 and u_1; the magnitude of $F(u_2) - F(u_1)$ varies as this movement takes place, and as we use this magnitude in the left side of Equation 8.45, we obtain the graph. To obtain the portion of the graphs for $x \leq 0$, we slide the phasor up toward the point z; by symmetry, we see that the magnitude of the phasor will be the same as before when the phasor was sliding toward $-z$.

Looking back at Figure 8.15, we see that the different graphs were obtained by choosing different values of D; Equation 8.46 says that these choices give different values for Δu, so that the arc length between u_2 and u_1 is different—thus a different graphs are obtained. Looking at the graph for $D = 790$ mm in Figure 8.15—the largest D value for the six graphs, and the one that gives a plot that approximates Fraunhofer behavior—we see that for this larger D value, Equation 8.46 predicts that Δu is smaller; thus phasors with a smaller arc length between u_1 and u_2 in Figure 8.16 move on the spiral to give approximately Fraunhofer behavior. On the other hand, when D gets smaller, then Δu gets bigger and so does the distance between u_1 and u_2 for the phasor; these phasors have the magnitude that varies rather dramatically as the phasor is moved along the Cornu spiral, and produces the interesting plots for the smaller values of D in Figure 8.15.

8.3.6 A rectangular aperture

At the beginning of this current Section 8.3, we started our work with a single aperture that was rectangular in shape. But it was convenient to put off the detailed description of the rectangular aperture until we had developed a useful mathematical structure, as well as the geometrical picture that we called the Cornu spiral. Then it was simpler to apply these concepts to first, an edge, and next, a slit. Now we are finally at the point where we can obtain the Fresnel diffraction from a rectangular slit, which we show in Figure 8.17.

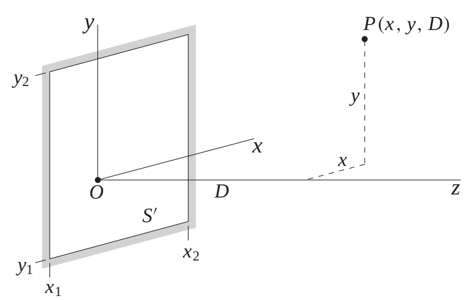

Figure 8.17

Before we perform the mathematical work to obtain the details of Fresnel diffraction by a rectangular aperture, we mention some aspects of this type of diffraction that we do not cover. One of the important apertures we do not study is a circular aperture, and the opposite, a circular obstruction. When Fresnel published his diffraction theory based on wave theory, many scientists believed strongly in the corpuscular, or particle, model of light—this model was shaped by Newton. When Poisson, one of the leading scientists in the early 1800s, read Fresnel's paper, he realized that the wave theory of Fresnel would predict that at the center of a shadow formed by a small circular obstruction there should be a point of light at the shadow center. Poisson thought this effect was absurd, and used the effect as proof that wave theory was false. However, Poisson communicated his ideas to Arago, another scientist. Arago made a 2 mm circular obstruction, and discovered that sure enough, there was a tiny spot of light at the shadow center. In fact, such a spot had been noticed some 50 years earlier, but the report had escaped widespread notice. Today, the spot is called Poisson's bright spot.

We now return to Fresnel diffraction by a rectangular aperture. One way to understand it is to regard it as the result of two slits at right angles to each other. But we shall look it at from the mathematical point-of-view with the equations that we have developed, namely, Equation 8.17, where we replace the coefficient $C(\lambda B/2)^2$ by $I_0/4$, as we showed in Equation 8.36. Or, perhaps it is simpler to look at Equation 8.43, and replace ∞ by v_2 and $-\infty$ by v_1:

$$\frac{I(P)}{I_0} = \frac{1}{2}\left\{[C(u_2) - C(u_1)]^2 + [S(u_2) - S(u_1)]^2\right\} \cdot \left\{[C(v_2) - C(v_1)]^2 + [S(v_2) - S(v_1)]^2\right\} \quad (8.47)$$

where (see Equations 8.12)

$$u_1 = \sqrt{\frac{2}{\lambda D}}\,(x_1 - x)\,, \qquad u_2 = \sqrt{\frac{2}{\lambda D}}\,(x_2 - x) \quad (8.48a)$$

$$v_1 = \sqrt{\frac{2}{\lambda D}}\,(y_1 - y)\,, \qquad v_2 = \sqrt{\frac{2}{\lambda D}}\,(y_2 - y) \quad (8.48b)$$

For numerical convenience, we illustrate the Fresnel diffraction pattern for a square aperture rather than a rectangular one: we choose $x_1 = -0.5$ mm, $x_2 = 0.5$ mm, for the x values; and $y_1 = -0.5$ mm, and $y_2 = 0.5$ mm for the y values. Then $\Delta x = \Delta y = 1.0$ mm. We choose $D = 156$ mm, and again choose the wavelength as $\lambda = 632.8$ nm. Substituting these values into the above Equations 8.47 and 8.48, and then using *Mathematica* to do the graphing, we obtain the three dimensional graph in Figure 7.17.

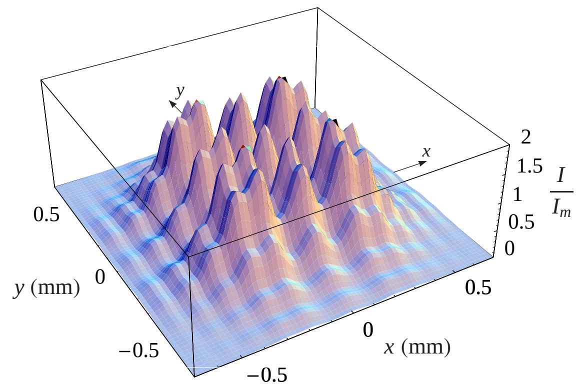

Figure 8.18

PROBLEMS

8.1 Assume $\lambda = 632.8 \times 10^{-6}$ mm and $D = 91$ mm. Also assume that $x' = y' = -0.5$ mm and $x = y = 5.4$ mm. Calculate ρ to six significant figures using Equations 8.3 and 8.5. The first result is the more accurate one for ρ, the second one is the approximate value; however, note that they do not differ by very much.

8.2 The Fresnel integrals $C(u)$ and $S(u)$ are defined in Equations 8.7. Focusing your attention on the integrand of $C(u)$, namely $\cos(\frac{\pi}{2}u'^2)$, over the interval from $u' = 0$ to $u' = 4$ find the values of u' for (a) the first eight zeros, (b) the first five maxima, and (c) the first four minima. Use these values as an aid to sketch (or graph) the function $\cos(\frac{\pi}{2}u'^2)$ from $u' = 0$ to $u' = 4$. The value of $C(4)$ is equal to the area under the curve that makes up this graph; note that the areas above the u' axis are positive and those below are negative.

8.3 Prove that both $C(u)$ and $S(u)$ are odd functions; that is, show that the expressions in Equation 8.20 are true.

8.4 In this problem, use the Cornu spiral in Figure 8.7. (a) First, use the Cornu spiral to graphically determine the magnitude of $F(1.5)$. One way to perform this determination is to line up the edge of a sheet of paper at the origin and the 1.5 ticmark on the spiral. Put marks on the paper edge to indicate the distance from the origin to the 1.5 ticmark. Then move the paper and place it along the bottom horizontal line (or the vertical one) where the ticmarks are numbered, and read off the distance in terms of these units between the marks on your paper edge; you have now found the value of the magnitude of $F(1.5)$. (b) Second, graphically determine $C(1.5)$. To make this reading, put marks on the paper edge that indicate the horizontal distance between the vertical axis and the 1.5 ticmark. Then, just as in part (a), move the paper and place it along the bottom horizontal line and read off the distance between the marks that you made. This value equals $C(1.5)$, and should be close to the value you read for C(1.5) in the table of Figure 8.5. (c) Third, determine $S(1.5)$ by using a method analogous to the part (b) method. You should obtain a value that is close to the one given for $S(1.5)$ in Figure 8.5. (d) Remembering that $F(u) = C(u) + i\,S(u)$, draw phasors to scale on a sheet of paper that represent $F(1.5) = C(1.5) + i\,S(1.5)$. Check the consistency of your results with $|F(1.5)| = \sqrt{[C(1.5)]^2 + [S(1.5)]^2}$.

8.5 Use the Cornu spiral in Figure 8.7 to find several values of I/I_0 that correspond to the single edge graphed in Figure 8.12; use the same values of $\lambda = 632.8$ nm, D = 160 mm, and $x_1 = -0.5$ mm. Evaluate I/I_0 for the values of $x = -1, -0.75, -0.5, -0.25, -0.1, 0,$ and 0.25 mm by first finding the corresponding values of u_1 using Equation 8.39, remembering to convert λ to mm before performing the calculations. After getting the u_1 quantities, and with $u_2 = \infty$, use the diagram in Figure 8.13 as a guide to obtain the magnitude of $F(\infty) - F(u_1)$ by applying the paper technique to the Cornu spiral in Figure 8.7. After obtaining this magnitude, Equation 8.42 says to square it, and then divide by two; you have now found I/I_0 (looking at Equations 8.40 and 8.41, as well as Equation 8.42, and reading the surrounding description will help in understanding what you are doing). The table is provided below as a guide, where mag represents the magnitude of $F(\infty) - F(u_1)$; that is, $|F(\infty) - F(u_1)|$:

x (mm)	u_1	mag	$I/I_0 = \text{mag}^2/2$
−1.00			
−0.75			
−0.50			
−0.25			
−0.10			
0.00			
0.25			

Chapter 8: Answers to Problems

8.1 91.3817 mm, 91.3825 mm

8.2 (a) 1, 1.73, 2.24, 2.65, 3, 3.32, 3.61, 3.87;
(b) 0, 2, 2.83, 3.46, 4;
(c) 1.41, 2.45, 3.16, 3.74

8.4 (a) 0.83, (b) 0.44, (c) 0.70

8.5 Your answers should be close to:
2.22, 0.14, 0.01;
1.11, 0.26, 0.035;
0, 0.71, 0.25;
−1.11, 1.64, 1.34;
−1.78, 1.28, 0.81;
−2.22, 1.50, 1.13;
−3.33, 1.34, 0.90
It is also useful to check your I/I_0 values with the graph in Figure 8.12.